AN ANABELIAN DEFINITION OF ABELIAN HOMOLOGY*

René GUITART

Abstract

Nous proposons une définition algébrique générale de l'homologie et la cohomologie qui sépare complètement le concept et les moyens de calcul, et qui est valide sans aucune hypothèse d'abélianité et de complétude. Nous montrons que dans le cas où le cadre de calcul est abélien complet et cocomplet, l'homologie en question redonne celle décrite par la définition abélienne usuelle.

In [7] we gave a non abelian or anabelian definition of classical satellites, related to exact squares [6]. Our aim here is to do the same for homology, and to provide a natural algebraic anabelian definition of homology and cohomology which reduces to the classical notion in the abelian case.

1 A first anabelian definition

At first, let us recall the well known process of Kan'extensions. For this notion see [8]. For the notations see [9].

If $K : \underline{A} \to \underline{B}$ is a functor and if $\underline{V}$ is a category, the left and right adjoints to functor $\underline{V}^K : \underline{V}^{\underline{B}} \to \underline{V}^{\underline{A}} : G \mapsto G \circ K$ are denoted (if they exist) by Lan_K and Ran_K (omitting the specification of $\underline{V}$) and are named left and right Kan extensions along K. So by definition we have

* *Cahiers Top. Géo. Diff. Cat.*, vol. XXXXVIII, 4 (2007), 261-269

the natural bijections

$$\mathrm{Nat}(Lan_K(F), G) \cong \mathrm{Nat}(F, G \circ K), \qquad < lan >$$

$$\mathrm{Nat}(G \circ K, F) \cong \mathrm{Nat}(G, Ran_K(F)), \qquad < ran >$$

and of course as left and right adjoints Lan_K and Ran_K are, respectively, right and left exact. But Lan_K is not necessarily left exact, and Ran_K is not necessarily right exact. So it could be useful to estimate how much Lan_K is not compatible with kernels and products, or how much Ran_K is not compatible with cokernel and sums: it will be specifically the job of homology and cohomology, as shown in our **definition 1** below (and as it is of course clear in the abelian case).

$$\begin{tikzcd} \underline{A} \arrow[r, "K"] \arrow[d, dotted, "F"'] \arrow[dr, phantom, "\Longrightarrow"] & \underline{B} \arrow[dl, dotted, "Lan_K F"] \\ \underline{V} & \end{tikzcd} \qquad \begin{tikzcd} \underline{A} \arrow[r, "K"] \arrow[d, dotted, "F"'] \arrow[dr, phantom, "\Longleftarrow"] & \underline{B} \arrow[dl, dotted, "Ran_K F"] \\ \underline{V} & \end{tikzcd}$$

If the given colimits or limits exist (for a given $F : \underline{A} \to \underline{V}$ and for each object B of $\underline{B}$) we know that :

$$Lan_K(F)(B) = \mathrm{colim}(K/B \overset{p_B}{\to} \underline{A} \overset{F}{\to} \underline{V}), \qquad [lan]$$

$$Ran_K(F)(B) = \lim(B/K \overset{q_B}{\to} \underline{A} \overset{F}{\to} \underline{V}), \qquad [ran]$$

where the canonical functor p_B (resp. q_B) is the K-co-shape (resp. the K-shape) of B, from the category K/B (resp. B/K) of objects of $\underline{A}$ over (resp. under) B toward the category $\underline{A}$. But the definition of Lan_K or Ran_K *does not suppose* that these limits exists, and a fortiori that these formulas make sense.

And we need also the functor $K^{\underline{V}} : \underline{A}^{\underline{V}} \to \underline{B}^{\underline{V}} : Q \mapsto K \circ Q$.

Now, in order to introduce our first definition of homology, for objects of $\underline{X}$ with values in $\underline{C}$, our basic datum $[J; L, R]$ consists of three functors.

The first one $J : \underline{M} \to \underline{X}$ describes the 'internal modeling datum" in $\underline{X}$. An object M of $\underline{M}$ is considered as a typical elementary or basic model for objects of $\underline{X}$, and each object X of $\underline{X}$ could be think as a glueing of such basic datum: an arrow $\alpha : J(M) \to X$ (resp. $\gamma : X \to J(M)$) in $\underline{X}$ is an elementary aspect (resp. co-aspect) of X. A classical case is $\underline{X} = CW - complexes$, with M the full subcategory of the Δ_n.

The two others $L, R : \underline{P} \to \underline{C}$ describe the "internal computing datum" in $\underline{C}$, and we think of (L, R) as an abstract calculus of presentation of objects of $\underline{C}$: if P is an object of $\underline{P}$, we think of $L(P)$ as an abstract quotient of P by $R(P)$. As explained in section 2, the classical abelian case is the one in which $C = \underline{Ab}$ and $P = EXA^n(\underline{Ab})$, etc., but clearly a lot of anabelian cases could be considerated naturally.

Definition 1 — *The homology and the cohomology with respect to the basic datum* $[J; L, R]$ *are the two functors* $H_\star[J; L, R]$ *and* $H^\star[J; L, R]$ *from* $\underline{X} \times \underline{C}^{\underline{M}}$ *to* $\underline{C}$ *which are given for* $X \in \underline{X}$ *and* $F : \underline{M} \to \underline{C}$ *by :*

$$H_\star[J; L, R](X, F) = Ran_{L^{\underline{M}}}(Lan_J \circ R^{\underline{M}})(F)(X),$$

$$\begin{tikzcd}
& \underline{M} \arrow[r, "J"] \arrow[d, dotted, "F"'] \arrow[dr, phantom, "\Longrightarrow"] & \underline{X} \arrow[dl, dotted, "Lan_J F"] \\
\underline{P} \arrow[r, "L"] \arrow[d, "R"'] & \underline{C} & \\
\underline{C} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P}^{\underline{M}} \arrow[r, "L^{\underline{M}}"] \arrow[d, "R^{\underline{M}}"'] \arrow[dr, phantom, "\Longleftarrow"] & \underline{C}^{\underline{M}} \arrow[d, "Ran_{L^{\underline{M}}}(Lan_J \circ R^{\underline{M}})"] \\
\underline{C}^{\underline{M}} \arrow[r, "Lan_J"'] & \underline{C}^{\underline{X}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{X} \times \underline{C}^{\underline{M}} \arrow[d, "H_\star"] \\
\underline{C}
\end{tikzcd}$$

$$H^\star[J; L, R](X, F) = Lan_{R^{\underline{M}}}(Ran_J \circ L^{\underline{M}})(F)(X),$$

$$\begin{tikzcd}
& \underline{M} \arrow[r, "J"] \arrow[d, dotted, "F"'] \arrow[dr, phantom, "\Longleftarrow"] & \underline{X} \arrow[dl, dotted, "Ran_J F"] \\
\underline{P} \arrow[r, "R"] \arrow[d, "L"'] & \underline{C} & \\
\underline{C} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P}^{\underline{M}} \arrow[r, "R^{\underline{M}}"] \arrow[d, "L^{\underline{M}}"'] \arrow[dr, phantom, "\Longrightarrow"] & \underline{C}^{\underline{M}} \arrow[d, "Lan_{R^{\underline{M}}}(Ran_J \circ L^{\underline{M}})"] \\
\underline{C}^{\underline{M}} \arrow[r, "Ran_J"'] & \underline{C}^{\underline{X}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{X} \times \underline{C}^{\underline{M}} \arrow[d, "H^\star"] \\
\underline{C}
\end{tikzcd}$$

Observations — 1 — In this definition the formulas for $H_\star$ and $H^\star$ provide, with respect to the calculus (L, R), an estimation at the level of X and from the point of view of F of the lack of left exactness of Lan_J and of the lack of right exactness of Ran_J. In this way we get an estimation, expressed by an object of $\underline{C}$, of how much X is far from $\underline{M}$.

2 — In this definition we saw neither additivity or abelianity nor existence of projective or injective resolution. So we did a step on the way to be free of contingent computations. Nevertheless the concept is not lost, because we are able to recover classical abelian things (section 2). But in order to be fully free of computation we have to discharge the definition 1 of the specifications of computing Ran and Lan (section 3).

2 Recovering the abelian case

Let us consider now the case where $\underline{C}$ is the category $\underline{\mathrm{Ab}}$ of abelian groups, where $\underline{P}$ is the category $EXA^n(\underline{Ab})$ of exact sequences of length $n+2$ in $\underline{Ab}$

$$E = (0 \leftarrow A_{n+2} \leftarrow A_{n+1} \leftarrow ...A_2 \leftarrow A_1 \leftarrow 0),$$

with $R(E) = A_1$ and $L(E) = A_{n+2}$.

In [7] it is proved that this datum EXA^n is obtained (up to isomorphism, but is it enough) by composition (by pullback) of the span (L, R) for EXA^1 with itself n times, and — that is the point — the used pullbacks are exact squares [6], a fact which implies that the n-th satellite is the nth-iteration of the first.

In this case $(EXA^n(\underline{Ab}))^{\underline{M}} \cong EXA^n(\underline{Ab}^{\underline{M}})$, and we can make use of [11] and of the universal description of satellites given in [7] (which unified as special cases the descriptions of [3], [5] and [10]) in order to say that the functors $Ran_{L^{\underline{M}}}(Lan_J \circ R^{\underline{M}})$ and $Lan_{R^{\underline{M}}}(Ran_J \circ L^{\underline{M}})$ are, respectively, the left (resp. right) satellites of order n of Lan_J (resp. of Ran_J).

But, as Lan_J (resp. Ran_J) is right (resp. left) exact, these satellites are also the corresponding derived functors $L_n(Lan_J)$ and $R^n(Ran_J)$ (see [4] for the definition of L_n and R^n), expressing how far from being compatible with kernels (resp. cokernels) is Lan_J (resp. is Ran_J). At this point, following [2], because of the exactness of the evaluation functor $E_X : \underline{C}^{\underline{X}} \to \underline{C}$, with $J_X : J/X \to \{*\}$ and ${}_XJ : X/J \to \{*\}$, we have :

$$H_\star[J; L, R](X, F) = H_\star[J_X; L, R](*, F \circ p_X),$$
$$H_\star[J; L, R](X, F) = H^\star[{}_XJ; L, R](*, F \circ q_X).$$

So, $H_\star$ (resp. $H^\star$) is a J-co-shape (resp. a J-shape) invariant of X.
In [2] it is also proved that in fact

$$L_n(Lan_J)(F)(X) \cong H_n^J(X, F),$$

where $H_n^J(X, F) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$ is the homology, introduced in [1], of the chain complex of abelian groups

$$... \to C_2(X, F) \stackrel{d_2}{\to} C_1(X, F) \stackrel{d_1}{\to} C_0[X, F) \stackrel{d_0}{\to} 0,$$

where

$$C_n(X, F) = \Sigma_{M_n \stackrel{m_n}{\to} M_{n-1} ... M_1 \stackrel{m_0}{\to} M_0; J(M_0) \stackrel{k}{\to} X} F(M_n),$$
$$d_n = \Sigma_{0 \le i \le n} (-1)^i s_n^i,$$

with

$$s_1^0 : (FM_1)_{(m_0;k)} \stackrel{Id}{\to} (FM_1)_{(M_1;k.J(m_0))},$$
$$s_1^1 : (FM_1)_{(m_0;k)} \stackrel{F(m_0)}{\to} (FM_0)_{(M_0;k)},$$

and so on.
We note that $H_0^J(X, F) = \mathrm{coker}(d_1) = Lan_J(F)(X)$.
And the dual of all that works for cohomology.

Finally in [1] it is shown that the various classical abelian homologies and cohomologies are examples of the $H_n^J(X, F)$ and the $H_J^n(X, F)$, and in this way we get in conclusion :

Theorem 1 *In the classical abelian cases the definition 1 in section 1 determines the classical theories of homology and cohomology.*

3 Homology and cohomology as limits and colimits, and the second definition

1 . For a basic datum $[J; L, R]$ as in definition 1, with a fixed functor $F : \underline{M} \to \underline{C}$, we consider as object a datum like

$$\alpha = [F; A, a; B, b]$$

with $A : \underline{M} \to \underline{P}$, $a : F{\circ}Id_{\underline{M}} \Rightarrow L{\circ}A$, $B : \underline{X} \to \underline{C}$, and $b : R{\circ}A \Rightarrow B{\circ}J$, as in the diagram

$$\begin{tikzcd}
\underline{M} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, Rightarrow, "a"'] & \underline{M} \arrow[r, "J"] \arrow[d, dotted, "A"'] & \underline{X} \arrow[d, "B"] \\
\underline{C} & \underline{P} \arrow[l, "L"] \arrow[r, "R"'] \arrow[ur, Rightarrow, "b"'] & \underline{C}
\end{tikzcd}$$

We define a morphism $\theta : \alpha \to \alpha'$ as a datum $\theta = (m, n)$ with

$$m : A \to A', \quad a' = (Lm)a, \quad n : B \to B', \quad (nJ)b = b'(Rm).$$

So we get a category of "homology" $\mathcal{H}_\star[J; L, R]_F$, with a forgetful functor toward $\underline{C}^{\underline{X}}$:

$$\Lambda_\star[J; L, R] : \mathcal{H}_\star[J; L, R]_F \to \underline{C}^{\underline{X}} : \alpha \mapsto B = \Lambda_\star(\alpha).$$

2. Dually, to describe a category of "cohomology" $\mathcal{H}^\star[J; L, R]_F$, with a forgetful functor toward $\underline{C}^{\underline{X}}$:

$$\Lambda^\star[J; L, R] : \mathcal{H}^\star[J; L, R]_F \to \underline{C}^{\underline{X}} : \alpha \mapsto B = \Lambda^\star(\alpha),$$

we consider an object like

$$\alpha = [F; A, a; B, b]$$

with $A : \underline{M} \to \underline{P}$, $a : R{\circ}A \Rightarrow F{\circ}Id_{\underline{M}}$, $B : \underline{X} \to \underline{C}$, and $b : B{\circ}J \Rightarrow L{\circ}A$, as in the diagram

$$\begin{tikzcd}
\underline{M} \arrow[r, equal] \arrow[d, "F"'] & \underline{M} \arrow[r, "J"] \arrow[d, dotted, "A"'] & \underline{X} \arrow[d, "B"] \arrow[dl, Rightarrow, "b"'] \\
\underline{C} & \underline{P} \arrow[l, "R"] \arrow[r, "L"'] \arrow[ul, Rightarrow, "a"] & \underline{C}
\end{tikzcd}$$

Theorem 2 *If we assume the convenient hypothesis of completeness and cocompleteness, we have*

$$H_\star[J;L,R](-,F) = \mathit{lim}\big(\Lambda_\star[J;L,R] : \mathcal{H}_\star[J;L,R]_F \to \underline{C}^{\underline{X}}\big),$$

$$H^\star[J;L,R](-,F) = \mathit{colim}\big(\Lambda^\star[J;L,R] : \mathcal{H}^\star[J;L,R]_F \to \underline{C}^{\underline{X}}\big).$$

For homology $H_\star$ we get, following definition 1 and formula $[ran]$,

$H_\star(-,F) = H_\star[J;L,R](-,F) = \lim_{(a:F\to L^{\underline{M}}(A);A)}\big(Lan_J(R^{\underline{M}}(A)\big)$, i.e. $H_\star(-,F) = \lim_{a:F\to L\circ A}\big(Lan_J(R\circ A)\big)$. But, by $< lan >$ for every $b : R\circ A \to B\circ J$, we have a factorization $b = (\beta J)\eta$ (with the universal $\eta : R\circ A \to Lan_J(R\circ A)\circ J$), and $Lan_J(R\circ A)$ dominates B in such a way that $\lim_{a:F\to L\circ A}\big(Lan_J(R\circ A)\big) = \lim_{a:F\to L\circ A;b:R\circ A\to B\circ J}\big(B\big)$, hence the formula for $H_\star$.

A completely dual argument works for $H^\star$. In fact, in this presentation it is clear that $H^\star[J;L,R](-,F) = \Big(H_\star[J^{\mathrm{op}};R^{\mathrm{op}},L^{\mathrm{op}}](-,F^{\mathrm{op}})\Big)^{\mathrm{op}}$.

Now we are ready for our second definition

Definition 2 — *Given as basic datum a couple $[I,J;L,R]$ of spans*

$$\underline{Y} \xleftarrow{\ I\ } \underline{M} \xrightarrow{\ J\ } \underline{X}\,, \qquad \underline{D} \xleftarrow{\ L\ } \underline{P} \xrightarrow{\ R\ } \underline{C}\,,$$

the homology *with respect to* $[I,J;L,R]$ with coefficients in *the functor* $F : \underline{Y} \to \underline{D}$ *is the functor* $H_\star(-,F) = H_\star[I,J;L,R](-,F) : \underline{X} \to \underline{C}$, *which is the projective limit*

$$H_\star[I,J;L,R](-,F) = \mathit{lim}\big(\Lambda_\star[I,J;L,R] : \mathcal{H}_\star[I,J;L,R]_F \to \underline{C}^{\underline{X}}\big),$$

of the functor $\Lambda_\star[I,J;L,R] : \mathcal{H}_\star[I,J;L,R]_F \to \underline{C}^{\underline{X}} : \alpha \mapsto B = \Lambda_\star(\alpha)$, *where* $\mathcal{H}_\star[I,J;L,R]_F$ *is the category with objects the* $\alpha = [F;A,a;B,b]$ *with* $A : \underline{M} \to \underline{P}$, $a : F\circ I \Rightarrow L\circ A$, $B : \underline{X} \to \underline{C}$, *and* $b : R\circ A \Rightarrow B\circ J$, *as in the diagram*

$$\begin{tikzcd}
\underline{Y} \arrow[d, "F"'] \arrow[dr, Rightarrow, "a"'] & \underline{M} \arrow[l, "I"'] \arrow[r, "J"] \arrow[d, dotted, "A"] & \underline{X} \arrow[d, "B"] \\
\underline{D} & \underline{P} \arrow[l, "L"] \arrow[r, "R"'] \arrow[ur, Rightarrow, "b"'] & \underline{C}
\end{tikzcd}$$

and with morphisms $\theta : \alpha \to \alpha'$ a datum $\theta = (m, n)$ with $m : A \to A'$, $a' = (Lm)a$, $n : B \to B'$, $(nJ)b = b'(Rm)$.

Clearly the dual definition works for cohomology.

Observations — 1 — The definition is given as a limit but does not need limits *inside* its expression.

2 — The fact of taking the limit becomes of secondary importance, the point beeing the description of the category $\mathcal{H}_\star[I, J; L, R]_F$ and the functor $\Lambda_\star[I, J; L, R] : \mathcal{H}_\star[I, J; L, R]_F \to \underline{C}^{\underline{X}}$.

3 — The point is to construct a comparison between two spans, the first one thought as an analysis of spaces living in $\underline{X}$, the second one thought as a calculus in $\underline{C}$. But at this level of abstraction the distinction between spaces and calculus disappeared.

4 — Of course, because of theorem 1 and theorem 2, this definition 2 again agrees with the classic descriptions of abelian homology and cohomology.

References

[1] M. André, *Méthode simpliciale en algèbre homologique et algèbre commutative*, S.L.N. 32, 1967.

[2] F. Bachmann, *Kategorische Homologietheorie un Spektralsequenzen*, Diss. 4277, ETH Zurich, 1969, 36p.

[3] D.A. Buchsbaum, *Homology and universality relative to a functor*, in S.L.N. vol. 61, 1968.

[4] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. press, 1956.

[5] J.L. Fisher, *The tensor product of functors, satellites, and derived functors*, J. of Al. 8, 277-294 (1968).

[6] R. Guitart, *Relations et carrés exacts*, Ann. Sci. math. Québec 4(1980), p. 103-125.

[7] R. Guitart et L. Van den Bril, *Calcul des satellites et présentations des bimodules à l'aide des carrés exacts, I et II*, Cahiers Top. Go. Diff., vol. XXIV-3 (1983), p. 299-330, et vol. XXIV-4 (1983), p. 333-369.

[8] D.M. Kan, *Adjoint functors*, trans. Am. math. Soc. 87, 294-329 (1958).

[9] S. MacLane, *Categories for the working mathematician*, Springer, 1970.

[10] F. Ulmer, *Satelliten und derivierte funktoren I*, Math. Zeitschr. 91 (1966).

[11] N. Yoneda, *On Ext and exact sequences*, J. fac. Sci. Tokyo sec I 8, 507-526 (1960).