Distributors at Work

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Contents

Abstract

The following notes originate from a course on *distributors* given by the author in June 2000 at TU Darmstadt on invitation of Thomas Streicher who has also prepared this introductory paper based on his lecture notes for which the author wants to express his gratitude.

It should be emphasised here that the aim of these notes is not to expand most recent research results but rather to give a well–motivated and easy to grasp introduction to the concept of distributors and its many applications justifying the title “Distributors at Work”.

1 Motivation

Analogies are useful in mathematics for generalising a class of well–known examples to a wider class of equally or even more useful structures. Category Theory is particularly well suited for this purpose which is no wonder as it has been developed for precisely this purpose. Well known examples of this method of “generalisation by analogy via category theory” are e.g.

- monoidal categories and enriched categories as generalisations of abelian categories, in particular categories of modules
- toposes as generalisations of the category of sets which via their internal language can be viewed as generalised universes of sets
- arbitrary Grothendieck fibrations as generalisations of the ‘family fibration’ $\mathrm{Fam}(\mathbf{C})$ over **Set** allowing for a development of category theory over an arbitrary base (topos).

In the following we concentrate on a generalisation of relations between sets to “relations between (small) categories” called *distributors*.

If A and B are sets then relations from A to B correspond to maps $A \to \mathcal{P}(B)$. If A and B are posets then relations from A to B may be defined as monotone maps $A \to {\downarrow}B$ where ${\downarrow}B$ is the poset of *downward closed* subsets of B ordered by set inclusion. Notice that this notion of relation between posets *generalises* the ordinary notion of relation between sets as for a set B considered as a discrete poset we have $\mathcal{P}(B) = {\downarrow}B$. Exploiting the cartesian closed nature of the category of posets and monotone maps we may define a relation from A to B simply as a monotone map $B^{\mathrm{op}} \times A \to \mathbf{2}$ where $\mathbf{2}$ is the 2 element lattice.[1] Such maps will be called “distributors between posets”.

[1]Notice that monotone maps $B^{op} \times A \to \mathbf{2}$ correspond to relations $R \subseteq B \times A$ such that

$$x \leq x' \wedge R(y,x) \wedge y' \leq y \Rightarrow R(y',x') \ .$$

Bearing in mind that posets are categories enriched over **2** and ordinary categories are "categories enriched over **Set**" we may define – by analogy with the case of posets – a relation between small categories **A** and **B** as a functor $\phi : \mathbf{B}^{op} \times \mathbf{A} \to \mathbf{Set}$ or, equivalently, as a functor $\mathbf{A} \to \widehat{\mathbf{B}}$ where $\widehat{\mathbf{B}} = \mathbf{Set}^{\mathbf{B}^{op}}$, the category of (**Set**–valued) presheaves on **B** which generalises $\downarrow B = \mathbf{2}^{B^{op}}$ for posets B in an obvious way. A functor $\phi : \mathbf{B}^{op} \times \mathbf{A} \to \mathbf{Set}$ will be called a *distributor* from **A** to **B** and we write $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ to express that ϕ is a distributor from **A** to **B**.

Whereas the categories of relations between sets and posets arise as Kleisli categories for monads $\mathcal{P}$ and $\downarrow$ on **Set** and **Poset**, respectively, from which it is obvious how to compose morphisms, this is not the case anymore for distributors between small categories as categories of presheaves even over a small category are themselves not small anymore. Instead we will describe composition of distributors between small categories via (left) Kan extension along Yoneda functors which we will discuss *en detail* in the next section. For distributors $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ and $\psi : \mathbf{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C}$ their composition $\psi\phi$ is given by $\mathrm{L}_{\mathbf{B}}\psi \circ \phi$ where $\mathrm{L}_{\mathbf{B}}\psi$ is the left Kan extension of $\psi : \mathbf{B} \to \widehat{\mathbf{C}}$ along the Yoneda functor $\mathsf{Y}_{\mathbf{B}} : \mathbf{B} \to \widehat{\mathbf{B}}$.

Notice that this definition of composition of distributors is in analogy with the definition of composition for relations and distributors between posets. If $\phi : A \to \mathcal{P}(B)$ and $\psi : B \to \mathcal{P}(C)$ then $\psi\phi = \mathrm{L}(\psi) \circ \phi$ where $\mathrm{L}(\psi) : \mathcal{P}(B) \to \mathcal{P}(C)$ is the unique *cocontinuous*[2] function g from $\mathcal{P}(B)$ to $\mathcal{P}(C)$ with $g \circ \{\cdot\} = \psi$. Similarly for composition of distributors between posets (just replace $\mathcal{P}$ by $\downarrow$). Now $\psi\phi = \mathrm{L}_{\mathbf{B}}\psi \circ \phi$ is a generalisation as $\mathrm{L}_{\mathbf{B}}(\psi) : \widehat{\mathbf{B}} \to \widehat{\mathbf{C}}$ is the unique (up to isomorphism) cocontinuous extension of ψ along $\mathsf{Y}_{\mathbf{B}}$.

2 Kan Extensions

We first introduce the key concept of left Kan extension.

Definition 2.1 *Let $F : \mathbf{A} \to \mathbf{X}$ and $G : \mathbf{A} \to \mathbf{Y}$ be functors. The functor F* has a left Kan extension along *G iff there is a functor $K : \mathbf{Y} \to \mathbf{X}$ together with a natural transformation $\eta : F \Rightarrow KG$ such that for all functors $K' : \mathbf{Y} \to \mathbf{X}$ and natural transformations $\varphi : F \Rightarrow K'G$ there is a unique*

[2]Here "cocontinuous" means preservation of colimits. This makes sense for functors in general. In the particular context it means that g preserves all suprema. There is the dual notion "continuous" meaning preservation of limits.

natural transformation $\psi : K \Rightarrow K'$ *with* $\varphi = \psi G \circ \eta$, *i.e. making the diagram*

$$\begin{array}{ccc} F & \overset{\eta}{\Longrightarrow} & KG \\ & {\scriptstyle \varphi}\searrow & \Downarrow {\scriptstyle \psi G} \\ & & K'G \end{array}$$

commute. ◊

Notice that left Kan extensions along F exist for all $G : \mathbf{A} \to \mathbf{Y}$ iff $\mathbf{X}^G : \mathbf{X}^{\mathbf{Y}} \to \mathbf{X}^{\mathbf{A}}$ has a left adjoint L_G, i.e.

$$\mathrm{Nat}(\mathrm{L}_G(F), K) \cong \mathrm{Nat}(F, KG)$$

naturally in F and K. Of course, one may also consider the dual concept of right Kan extension $\mathbf{X}^G \dashv \mathrm{R}_G$ satisfying

$$\mathrm{Nat}(KG, F) \cong \mathrm{Nat}(K, \mathrm{R}_G(F))$$

naturally in in F and K. We leave it as an exercise to state the universal properties of $\mathrm{R}_G(F)$ analogous to Definition 2.1. Notice, however, that right Kan extensions will not play any prominent role in the development of the theory of distributors in these notes.

Notation In the following small categories will be denoted by $\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$ whereas arbitrary, possibly big categories will be denoted by $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \ldots$.

Next we recall a theorem of Kan guaranteeing the existence of left Kan extensions under very mild assumptions.

Theorem 2.1 *Let* $F : \mathbf{A} \to \mathbf{X}$ *and* $G : \mathbf{A} \to \mathbf{Y}$ *be functors. If* $\mathbf{A}$ *is small and* $\mathbf{X}$ *is cocomplete then there exists a Kan extension* $\mathrm{L}_G(F)$ *of* F *along* G*. Moreover, if* G *is full and faithful then* $\mathrm{L}_G(F) \circ G \cong F$*.*

Proof. For $Y \in \mathbf{Y}$ define $\mathrm{L}_G(Y)$ as $\mathrm{Colim}_{G(A) \to Y} F(A)$, the colimit of

$$G{\downarrow}Y \xrightarrow{\ \partial_0\ } \mathbf{A} \xrightarrow{\ F\ } \mathbf{X} \quad ,$$

and η_A as the component of the colimiting diagram at (A, id_{GA}).

If G is full and faithful then $G{\downarrow}G(A) \simeq \mathbf{A}{\downarrow}A$ and, therefore, η_A is an isomorphism. □

Of course, L_G preserves all colimits as it is a left adjoint. However, in general $\mathrm{L}_G(F)$ does not have any particular good preservation properties as can be seen when considering Kan extension along $\mathrm{id}_{\mathbf{A}}$ where $\mathrm{L}_{\mathrm{id}_{\mathbf{A}}}(F)$ is as bad (or good) as F itself because $\mathrm{L}_{\mathrm{id}_{\mathbf{A}}}(F) \cong F$.

However, much more can be said for the special case of Kan extension along Yoneda functors. We will write $\mathrm{L}_{\mathbf{A}}$ as an abbreviation for $\mathrm{L}_{\mathsf{Y}_{\mathbf{A}}}$, i.e. left Kan extension along the Yoneda functor $\mathsf{Y}_{\mathbf{A}} : \mathbf{A} \to \widehat{\mathbf{A}}$.

First let us recall some basic facts about presheaf toposes. For $P \in \widehat{\mathbf{A}}$ its category of elements $\mathbf{Elts}(P)$ is defined as follows: objects are pairs (A, a) with $a \in P(A)$, morphisms from (A, a) to (B, b) are morphisms $f : A \to B$ in $\mathbf{A}$ with $a = P(f)(b)$ and composition is inherited from $\mathbf{A}$. Notice that by the Yoneda lemma $\mathbf{Elts}(P)$ is isomorphic to $\mathsf{Y}_{\mathbf{A}}{\downarrow}P$. The colimit of

$$\mathsf{Y}_{\mathbf{A}}{\downarrow}P \xrightarrow{\partial_0} \mathbf{A} \xrightarrow{\mathsf{Y}_{\mathbf{A}}} \widehat{\mathbf{A}}$$

is P where the component of the colimiting cone at $(A, a : \mathsf{Y}_{\mathbf{A}}(A) \to P)$ is a. Thus, every presheaf is *canonically the colimit of its approximating representable objects.*

Theorem 2.2 *Let* $\mathbf{A}$ *be small and* $\mathbf{X}$ *be cocomplete. Then for every functor* $F : \mathbf{A} \to \mathbf{X}$ *its left Kan extension* $\mathrm{L}_{\mathbf{A}}(F) = \mathrm{L}_{\mathsf{Y}_{\mathbf{A}}}(F)$ *is cocontinuous and has a right adjoint, namely the exponential transpose of the functor* $\mathbf{X}(F(-_1), -_2) : \mathbf{A}^{op} \times \mathbf{X} \to \mathbf{Set}$.

Proof. The Kan extension $\mathrm{L}_{\mathbf{A}}(F)$ preserves colimits due to its construction via colimits (as explained in the proof of Theorem 2.1) and the fact that every $P \in \widehat{\mathbf{A}}$ is canonically the colimit of its approximating representable presheaves.

Thus, by the Adjoint Functor Theorem $\mathrm{L}_{\mathbf{A}}(F)$ has a right adjoint R which must look as follows

$$R(X)(I) \cong \widehat{\mathbf{A}}(\mathsf{Y}_{\mathbf{A}}(I), R(X)) \cong \mathbf{X}(\mathrm{L}_{\mathbf{A}}(F)(\mathsf{Y}_{\mathbf{A}}(I)), X) \cong \mathbf{X}(F(I), X)$$

where the last isomorphism is induced by the isomorphism $\mathrm{L}_{\mathbf{A}}(F) \circ \mathsf{Y}_{\mathbf{A}} \cong F$ following from Theorem 2.1 because $\mathsf{Y}_{\mathbf{A}}$ is full and faithful. □

Historical Note. The previous theorem was known in special cases already before D. Kan proved it. Actually, his motivation for the theorem was the following well–known situation.

Consider the topos $\widehat{\Delta}$ of simplicial sets where Δ is the category of finite nonempty ordinals which embeds into the category $\mathbf{Sp}$ of topological spaces via a functor $F : \Delta \to \mathbf{Sp}$ sending 1 to a point, 2 to a line segment, 3 to a triangle etc. *Singular homology* for spaces is defined via the functor

$$\mathsf{Sing} : \mathbf{Sp} \to \widehat{\Delta} : X \mapsto [n \mapsto \mathbf{Sp}(F(n), X)]$$

and it was observed by Milnor that Sing has a left adjoint $\mathsf{Real} = \mathrm{L}_\Delta(F)$ providing *geometric realisation* of simplicial complexes.

In order to understand more abstractly situations like these D. Kan introduced the concepts of adjoint functors and Kan extension.

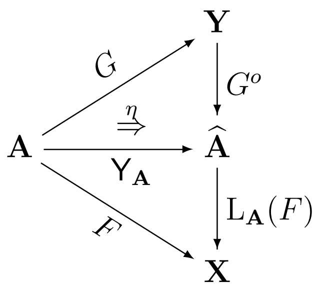
Next we show how Kan extensions along functors between small categories can be reduced to Kan extensions along Yoneda.

For a functor $G : \mathbf{A} \to \mathbf{Y}$ with $\mathbf{A}$ small and $\mathbf{Y}$ locally small let $G^o : \mathbf{Y} \to \widehat{\mathbf{A}}$ be the exponential transpose of $\mathbf{Y}(G(-_1), -_2) : \mathbf{A}^{op} \times \mathbf{Y} \to \mathbf{Set}$ sending Y to $\mathbf{Y}(G(-), Y) \in \widehat{\mathbf{A}}$. Let $\eta : \mathsf{Y}_\mathbf{A} \Rightarrow G^o G$ be the natural transform with

$$(\eta_A)_{A'} : \mathbf{A}(A', A) \to \mathbf{Y}(GA', GA) : u \mapsto G(u)$$

for $A, A' \in \mathbf{A}$. In the next section we will see that if $\mathbf{Y}$ is a small category then G^o is the distributor right adjoint to $\mathsf{Y}_\mathbf{B} \circ G$ with η being the unit of the adjunction.

Theorem 2.3 *Let* $\mathbf{A}$ *be a small category and* $G : \mathbf{A} \to \mathbf{Y}$ *be a functor. Then for every functor* $F : \mathbf{A} \to \mathbf{X}$ *with* $\mathbf{X}$ *cocomplete the left Kan extension* $\mathrm{L}_G(F)$ *is given by* $\mathrm{L}_\mathbf{A}(F) \circ G^o$ *and its universal property is exhibited by the natural transformation* $\mathrm{L}_\mathbf{A}(F)\eta$ *as shown in the diagram*



Proof. Notice that $\mathbf{Elts}(G^o(Y)) = \mathbf{Elts}(\mathbf{Y}(G(-),Y))$ is canonically isomorphic to $G{\downarrow}Y$ from which it follows that $\mathrm{L}_{\mathbf{A}}(F)(G^o(B)) = \mathrm{L}_G(F)(B)$. The universal property of $\mathrm{L}_{\mathbf{A}}(F)\eta$ follows from the fact that $\mathrm{L}_{\mathbf{A}}(F)(\eta_A)$ is the leg of the colimiting cone for

$$G{\downarrow}G(A) \xrightarrow{\partial_0} \mathbf{A} \xrightarrow{F} \mathbf{X}$$

at $(A, \mathrm{id}_{G(A)})$. □

Notice that Theorem 2.3 may be used to prove some preservation property of Kan extensions $\mathrm{L}_G(F)$ as if G^o and $\mathrm{L}_{\mathbf{A}}(F)$ have the preservation property under consideration then so has their composite $\mathrm{L}_G(F) = \mathrm{L}_{\mathbf{A}}(F) \circ G^0$.

Another consequence of Theorem 2.3 is the following.

Corollary 2.4 *Let $G : \mathbf{A} \to \mathbf{Y}$ with $\mathbf{A}$ small. Then G is full and faithful iff $\eta : \mathsf{Y}_{\mathbf{A}} \Rightarrow \mathrm{L}_G(\mathsf{Y}_{\mathbf{A}}) \circ G$ is an isomorphism. Thus, G is full and faithful iff $\eta : F \Rightarrow \mathrm{L}_G(F) \circ G$ is canonically isomorphic for all $F : \mathbf{A} \to \mathbf{X}$ with $\mathbf{X}$ cocomplete.*

Proof. As $\mathrm{L}_{\mathbf{A}}(\mathsf{Y}_{\mathbf{A}})$ is (isomorphic to) the identity on $\widehat{\mathbf{A}}$ from Theorem 2.3 we get that $\mathrm{L}_G(\mathsf{Y}_{\mathbf{A}}) = G^o$ and, therefore, that $\eta : \mathsf{Y}_{\mathbf{A}} \Rightarrow \mathrm{L}_G(\mathsf{Y}_{\mathbf{A}}) \circ G$ is an isomorphism iff $\eta : \mathsf{Y}_{\mathbf{A}} \Rightarrow G^0 G$ is an isomorphism which in turn is equivalent to G being full and faithful. Thus, the functor G is full and faithful if $\eta : F \Rightarrow \mathrm{L}_G(F) \circ G$ is an isomorphism for all $F : \mathbf{A} \to \mathbf{X}$ with $\mathbf{X}$ cocomplete. The reverse implication is immediate by Theorem 2.1. □

3 Distributors

In this section we define the bicategory **Dist** of distributors between small categories and study its basic properties.

Definition 3.1 *Let $\mathbf{A}$ and $\mathbf{B}$ be small categories. A* distributor *from $\mathbf{A}$ to $\mathbf{B}$ is a functor $\phi : \mathbf{B}^{op} \times \mathbf{A} \to \mathbf{Set}$. We write $\phi : \mathbf{A} \not\longrightarrow \mathbf{B}$ to indicate that ϕ is a distributor from $\mathbf{A}$ to $\mathbf{B}$.* ◊

Notice that there is a natural 1–1–correspondence between distributors from $\mathbf{A}$ to $\mathbf{B}$ and functors $\mathbf{A} \to \widehat{\mathbf{B}}$ as **Cat** is cartesian closed.[3] We write $\hat{\phi}$ for the exponential transpose of a distributor ϕ.

[3]We have a slight preference for the first view as it is slightly more general in the sense that it does not require the existence of $\widehat{\mathbf{B}}$. For example the representation of relations

Let $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ be a distributor between small categories and $x \in \phi(B, A)$. If $a : A \to A'$ in $\mathbf{A}$ then we write ax for $\phi(B, a)(x) \in \phi(B, A')$ and if $b : B' \to B$ in $\mathbf{B}$ then we write xb for $\phi(b, A)(x) \in \phi(B', A)$. Thus, distributors can most naturally be considered as families of sets on which simultaneously $\mathbf{A}$ acts from the left and $\mathbf{B}$ acts from the right and these two actions commute with each other, i.e. are related by the law

$$(ax)b = a(xb)$$

for $x \in \phi(B, A)$, $a : A \to A'$ and $b : B' \to B$.

Notice that distributors are related to ordinary *bimodules* in the following way. Rings can be considered as categories with one object enriched over abelian groups. Enriched distributors from A to B then are nothing else but an abelian group M on which A acts from the left and B acts from the right and these actions commute with each other. This analogy between distributors and bimodules will play a guiding role subsequently.

Let $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ and $\psi : \mathbf{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C}$ be distributors. Recall from the introduction that we intend to compose distributors in the following way

$$\psi\phi = \mathrm{L}_{\mathbf{B}}(\hat{\psi}) \circ \phi$$

i.e. as ϕ followed by the left Kan extension of $\hat{\psi}$ along the Yoneda functor $\mathsf{Y}_{\mathbf{B}}$. Thus, according to Theorem 2.1 $(\psi\phi)(C, A)$ is given by the colimit of the diagram

$$\mathbf{Elts}(\phi(-, A)) \xrightarrow{\partial_0} \mathbf{B} \xrightarrow{\psi(C,-)} \mathbf{Set}$$

as colimits in $\widehat{\mathbf{C}}$ are computed pointwise. Thus, we have

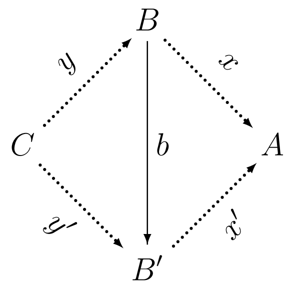
$$(\psi\phi)(C, A) = \big(\coprod_{B \in \mathbf{B}} \psi(C, B) \times \phi(B, A)\big)_{/\sim}$$

where $\sim$ is the least equivalence relation with $(y', x') \sim (y, x)$ if there is a morphism b in $\mathbf{B}$ with

$$y' = by \qquad \text{and} \qquad x'b = x$$

from A to B as functions $B \times A \to 2$ is simpler than the representation as functions $A \to \mathcal{P}(B)$ as the former does not require the existence of powersets which in general do not exist within e.g. the category of countable sets (unless B is finite) whereas functions $B \times A \to 2$ do stay within countable sets.

as indicated in the diagram



where the dotted arrows stand for “fake arrows”, i.e. elements of $\phi(B, A)$, $\phi(B', A)$, $\psi(C, B)$ and $\psi(C, B')$, respectively.

Thus, $(\psi\phi)(C, A)$ consists of the *connected components* of the comma category of the cospan

$$\mathbf{Elts}(\psi(C,-)) \xrightarrow{\int \psi(C,-)} \mathbf{B} \xleftarrow{\int \phi(-,A)} \mathbf{Elts}(\phi(-,A))$$

where $\int \phi(-, A)$ and $\int \psi(C, -)$ are the fibration and cofibration obtained from $\phi(-, A)$ and $\psi(C, -)$, respectively, via the Grothendieck construction. As composition of distributors requires to have connected components available in order to develop the basic theory of distributors in other universes $\mathbf{U}$ than $\mathbf{Set}$ such $\mathbf{U}$ not only have to have finite limits but also well–behaved coequalisers[4], i.e. $\mathbf{U}$ has to be *exact*.

Definition 3.2 *The bicategory* **Dist** *of distributors is defined as follows.*

The 0*–cells of* **Dist** *are the small categories. The* 1*–cells from* **A** *to* **B** *are the distributors from* **A** *to* **B***. A* 2*–cell between distributors* $\phi, \phi' : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{B}$ *is a natural transformation from* ϕ *to* ϕ'*.*

If $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{B}$ *and* $\psi : \mathbf{B} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{C}$ *are distributors then their composite* $\psi\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{C}$ *is defined (as discussed already above) as follows*

$$(\psi\phi)(C, A) = \big(\coprod_{B \in \mathbf{B}} \psi(C, B) \times \phi(B, A)\big)_{/\sim}$$

[4]Let $\mathbf{C}$ be a category with pullbacks. Then for internal graphs in $\mathbf{C}$ the connected components functor Π_0 is defined as the left adjoint to the inclusion $\mathsf{disc} : \mathbf{C} \to \mathrm{Graph}(\mathbf{C})$ sending $C \in \mathbf{C}$ to the span $(\mathrm{id}_C, \mathrm{id}_C)$. Obviously, $\mathbf{C}$ admits a Π_0 if and only if $\mathbf{C}$ has coequalizers. Of course, the connected components of an internal category are the connected components of its underlying graph.

where $\sim$ *is the least equivalence relation with* $(y', x') \sim (y, x)$ *if there is a morphism* b *in* $\mathbf{B}$ *with* $y' = by$ *and* $x'b = x$. *Composition of distributors is not associative on the nose but, rather, associativity only holds up to the isomorphism*

$$\alpha : (\theta\psi)\phi \overset{\sim}{\Longrightarrow} \theta(\psi\phi)$$

sending $[((z, y), x)]_\sim$ *to* $[(z, (y, x))]_\sim$. *For* $\mathbf{A}$ *the identity* 1*–cell* $\mathrm{id}_\mathbf{A} : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\ \mathbf{A}$ *is given by* $\mathrm{Hom}_\mathbf{A}(-, -) : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$. *They are left and right neutral only up to the isomorphisms*

$$\lambda : \phi\mathrm{id} \overset{\sim}{\Longrightarrow} \phi \qquad \rho : \mathrm{id}\phi \overset{\sim}{\Longrightarrow} \phi$$

sending $[(x, f)]$ *and* $[(f, x)]$ *to* fx *and* xf, *respectively.* $\diamondsuit$

The verification that these data satisfy the laws of a bicategory is tedious but straightforward. In a sense it is only the non–associativity of cartesian product (in **Set**) that renders composition of distributors not strictly associative.

However, if we define composition of distributors via left Kan extension then there is a nice conceptual proof of associativity. Let $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\ \mathbf{B}$, $\psi : \mathbf{B} \longrightarrow\!\!\!\!\!\!+\ \mathbf{C}$ and $\theta : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\ \mathbf{D}$ be distributors then

$$(\theta\psi)\phi = \mathrm{L}_\mathbf{B}(\mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \psi) \circ \phi \cong \mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \mathrm{L}_\mathbf{B}(\hat{\psi}) \circ \phi = \theta(\psi\phi)$$

as it already holds that

$$\mathrm{L}_\mathbf{B}(\mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \psi) \cong \mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \mathrm{L}_\mathbf{B}(\hat{\psi})$$

as both functors are cocontinuous and isomorphic when restricted along $\mathsf{Y}_\mathbf{B}$ as $\mathrm{L}_\mathbf{B}(\mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \psi) \circ \mathsf{Y}_\mathbf{B} \cong \mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \psi \cong \mathrm{L}_\mathbf{C}(\hat{\theta}) \circ \mathrm{L}_\mathbf{B}(\psi) \circ \mathsf{Y}_\mathbf{B}$.

4 Tensor and Hom

Composing distributors via Kan extension is a fruitful point of view as it suggests the following notion of *"tensor product"*.

Definition 4.1 *Let* $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\ \mathbf{B}$ *be a distributor and* $F : \mathbf{B} \to \mathbf{X}$ *be a functor with* $\mathbf{X}$ *a cocomplete category. Then their* tensor product $F\phi$ *is defined as* $\mathrm{L}_\mathbf{B}(F) \circ \hat{\phi}$. $\diamondsuit$

It is easily checked that the tensor product is associative up to isomorphism in the sense that

$$F(\phi\psi) \cong (F\phi)\psi \qquad\qquad (GF)\phi \cong G(F\phi)$$

for distributors ψ and cocontinuous functors G (between cocomplete categories).

A particular instance of this quite general notion of tensor product is considered in vol.1 of [Bor](p.128) where for $F : \mathbf{C} \to \mathbf{Set}$ and $G : \mathbf{C}^{op} \to \mathbf{Set}$ he defines $F \otimes G$ as $F\psi$ (in the sense of Definition 4.1) where $\psi : \mathbf{1} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C}$ is the distributor corresponding to G via

$$\frac{\dfrac{G : \mathbf{C}^{op} \to \mathbf{Set}}{\psi : \mathbf{C}^{op} \times \mathbf{1} \to \mathbf{Set}}}{\psi : \mathbf{1} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C}}$$

Equivalently, one may define $F \otimes G$ as $\phi\psi : \mathbf{1} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{1}$ where $\phi : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{1}$ is the distributor corresponding to F via

$$\frac{\dfrac{F : \mathbf{C} \to \mathbf{Set}}{\phi : \mathbf{1}^{op} \times \mathbf{C} \to \mathbf{Set}}}{\phi : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{1}}$$

which makes sense as $\phi\psi : \mathbf{1}^{op} \times \mathbf{1} \to \mathbf{Set}$ is (essentially) a set.[5]

Notice that composition of distributors

$$\mathbf{Dist}(\mathbf{B}, \mathbf{C}) \times \mathbf{Dist}(\mathbf{A}, \mathbf{B}) \to \mathbf{Dist}(\mathbf{A}, \mathbf{C}) : (\psi, \phi) \mapsto \psi\phi$$

is cocontinuous. Therefore, for $\phi \in \mathbf{Dist}(\mathbf{A}, \mathbf{B})$ the cocontinuous functor

$$(-)\phi : \mathbf{Dist}(\mathbf{B}, \mathbf{C}) \to \mathbf{Dist}(\mathbf{A}, \mathbf{C})$$

between presheaf toposes by the adjoint functor theorem has a right adjoint

$$\mathrm{Hom}_{\mathbf{A}}(\phi, -) : \mathbf{Dist}(\mathbf{A}, \mathbf{C}) \to \mathbf{Dist}(\mathbf{B}, \mathbf{C})$$

giving rise to a natural 1–1–correspondence

$$\frac{\psi\phi \Rightarrow \theta}{\psi \Rightarrow \mathrm{Hom}_{\mathbf{A}}(\phi, \theta)}$$

[5] It seems to be appropriate to think of the composite $\psi\phi$ as sort of a tensor product as one factors by the least equivalence relation rendering the pairs (by, x) and (y, xb) equal in analogy to the construction of tensor products for abelian groups or modules.

and using Yoneda one easily checks that

$$\mathrm{Hom}_{\mathbf{A}}(\phi,\theta)(C,B) \cong \mathrm{Nat}(\phi(B,-),\theta(C,-)) \ \ .$$

Similarly, there is a right adjoint $\mathrm{Hom}^{\mathbf{A}}(\psi,-)$ to $\psi(-)$ giving rise to a natural 1–1–correspondence

$$\frac{\psi\phi \Rightarrow \theta}{\phi \Rightarrow \mathrm{Hom}^{\mathbf{C}}(\psi,\theta)}$$

and using Yoneda one easily checks that

$$\mathrm{Hom}^{\mathbf{C}}(\psi,\theta)(B,A) = \mathrm{Nat}(\psi(-,B),\theta(-,A)) \ \ .$$

We close this section by observing that for every small category $\mathbf{A}$ we have that $\mathbf{A}{\downarrow}\mathbf{Dist}$ is enriched in $\mathbf{Dist}$ in the following way: for distributors $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$, $\psi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C}$ and $\theta : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{D}$ there is a composition map

$$\frac{\mathrm{Hom}_{\mathbf{A}}(\psi,\theta)\mathrm{Hom}_{\mathbf{A}}(\phi,\psi) \Rightarrow \mathrm{Hom}_{\mathbf{A}}(\phi,\theta)}{\mathrm{Hom}_{\mathbf{A}}(\psi,\theta)\mathrm{Hom}_{\mathbf{A}}(\phi,\psi)\phi \Rightarrow \mathrm{Hom}_{\mathbf{A}}(\psi,\theta)\psi \Rightarrow \theta}$$

where the two natural transformations below the line are induced by the counits $\mathrm{Hom}_{\mathbf{A}}(\phi,\psi)\phi \Rightarrow \psi$ and $\mathrm{Hom}_{\mathbf{A}}(\psi,\theta)\psi \Rightarrow \theta$ of the Tensor–Hom–Adjunction.

5 Duality for Distributors and Right Adjoints

It is well known that by dualisation a natural transformation

$$\varphi : f \to g : \mathbf{A} \to \mathbf{B}$$

between functors turns into a natural transformation

$$\varphi^{\mathrm{op}} : g^{\mathrm{op}} \to f^{op} : \mathbf{B}^{op} \to \mathbf{A}^{op}$$

i.e. $(-)^{op}$ reverts 1–cells and 2–cells. However, for relation the situation is quite different. If $R \subseteq S : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ then $R^o \subseteq S^o : B \longrightarrow\!\!\!\!\!\!\!+\;\; A$. A similar phenomenon holds for $\mathbf{Dist}$ which is no surprise as it is a generalisation of relations.

Definition 5.1 *Let $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!+\ \mathbf{B}$ be a distributor then let $\phi^\sigma : \mathbf{A}^{op} \longrightarrow\!\!\!\!\!+\ \mathbf{B}^{op}$ be the distributor given by*

$$\begin{array}{c} \phi : \mathbf{A} \longrightarrow\!\!\!\!\!+\ \mathbf{B} \\ \hline \phi : \mathbf{B}^{op} \times \mathbf{A} \longrightarrow \mathbf{Set} \\ \hline \mathbf{A} \times \mathbf{B}^{op} \longrightarrow \mathbf{Set} \\ \hline \phi^\sigma : (\mathbf{A}^{op})^{op} \times \mathbf{B}^{op} \longrightarrow \mathbf{Set} \\ \hline \phi^\sigma : \mathbf{B}^{op} \longrightarrow\!\!\!\!\!+\ \mathbf{A}^{op} \end{array}$$

and for $\varphi : \phi \Rightarrow \psi : \mathbf{A} \longrightarrow\!\!\!\!\!+\ \mathbf{B}$ let $\varphi^\sigma : \phi^\sigma \Rightarrow \psi^\sigma : \mathbf{B}^{op} \longrightarrow\!\!\!\!\!+\ \mathbf{A}^{op}$ be given by $\varphi^\sigma_{A,B} = \varphi_{B,A}$.

It is straightforward to check that

$$(\phi^\sigma)^\sigma = \phi \qquad (\psi\phi)^\sigma = \phi^\sigma\psi^\sigma \qquad (\mathrm{id}_{\mathbf{A}})^\sigma = \mathrm{id}_{\mathbf{A}^{op}}$$

as expected. For this reason cocontinuity of composition of distributors follows from cocontinuity in one argument. Moreover, left and right Hom as discussed above are interrelated in the following way

$$\begin{array}{c} \psi \Rightarrow \mathrm{Hom}_{\mathbf{A}}(\phi, \theta) \\ \hline \psi\phi \Rightarrow \theta \\ \hline (\psi\phi)^\sigma \Rightarrow \theta^\sigma \\ \hline \phi^\sigma\psi^\sigma \Rightarrow \theta^\sigma \\ \hline \psi^\sigma \Rightarrow \mathrm{Hom}^{\mathbf{A}^{op}}(\phi^\sigma, \theta^\sigma) \\ \hline \psi \Rightarrow \mathrm{Hom}^{\mathbf{A}^{op}}(\phi^\sigma, \theta^\sigma)^\sigma \end{array}$$

from which it follows that one can define one from the other using the duality $(-)^\sigma$.

Next we use the duality $(-)^\sigma$ to construct right adjoint distributors to functors between small categories considered as distributors.

Definition 5.2 *For a functor $f : \mathbf{A} \to \mathbf{B}$ between small categories there are associated distributors*

$$\phi_f = \mathbf{B}(-, f(-)) : \mathbf{A} \longrightarrow\!\!\!\!\!+\ \mathbf{B}$$

and

$$\phi^f = (\phi_{f^{op}})^\sigma : \mathbf{B} \longrightarrow\!\!\!\!\!+\ \mathbf{A}$$

where ϕ_f is called "f considered as a distributor" and ϕ^f is called "its right adjoint distributor.

Notice that explitating the definition of ϕ^f gives $\phi^f(A,B) = \mathbf{B}(f(A),B)$. The following lemma justifies the terminology "ϕ^f is the right adjoint distributor of ϕ_f".

Lemma 5.1 *For a functor $f : \mathbf{A} \to \mathbf{B}$ between small categories we have*

$$\phi_f \dashv \phi^f$$

i.e. ϕ^f is right adjoint to ϕ_f in the bicategory **Dist**.

Proof. The unit and counit of the adjunction are given by

$$\eta : \mathrm{id}_{\mathbf{A}} \Rightarrow \phi^f \phi_f : a \mapsto f(a)$$

and

$$\varepsilon : \phi_f \phi^f \Rightarrow \mathrm{id}_{\mathbf{B}} : [(b', b)]_\sim \mapsto b'b$$

respectively.

The validity of the required triangle equalities is easy to check. □

Actually, there is the following strengthening of Lemma 5.1.

Theorem 5.2 *A distributor $\phi : \mathbf{A} \to\!\!\!\!\!\!+\; \mathbf{B}$ has a right adjoint iff $\hat{\phi} : \mathbf{A} \to \widehat{\mathbf{B}}$ factors through the Cauchy completion of* $\mathbf{B}$, *i.e. the full subcategory of $\widehat{\mathbf{B}}$ on retracts of representable presheaves.*

Proof. See Volume 1 of [Bor]. □

From this it follows that **A** and **B** are equivalent in the bicategory **Dist** iff $\widehat{\mathbf{A}} \simeq \widehat{\mathbf{B}}$. In general, such an equivalence is not induced by a functor. However, it can be shown that **A** and **B** are equivalent in **Dist** iff their *Cauchy completions*, i.e. splitting of idempotents, are equivalent as categories (*cf.* [Bor]).

In the remainder of this section we demonstrate how duality may be fruitfully applied to the study of categories of relations. In [FrSc] one can find the definition of an *allegory* providing an axiomatic account of categories of relations. Essentially an allegory $\mathcal{A}$ is a poset enriched category with an involution $(-)^{op}$ where the hom–sets are meet semi–lattices, $(-)^{op}$ preserves this structure and, moreover, the law of

(Modularity) $\quad SR \cap T \leq (S \cap TR^\circ)R$

holds.

In [FrSc] maps were characterised as those morphisms R such that

$$RR^{\mathrm{op}} \leq 1 \qquad \text{and} \qquad 1 \leq R^{\mathrm{op}}R$$

hold. The first requirement expresses single–valuedness and the second one expresses totality of R. Though not stated in [FrSc] maps in an allegory can be characterised as follows.

Theorem 5.3 *A morphism $R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ in an allegory $\mathcal{A}$ is a map iff R has a right adjoint S, i.e. $S : B \longrightarrow\!\!\!\!\!\!\!+\;\; A$ with $1 \leq SR$ and $RS \leq 1$. Moreover,, the right adjoint of R is R^{op} provided it exists.*

Proof. If R is a map then obviously R^{op} is the right adjoint of R.

For the reverse direction suppose that $R \dashv S$, i.e. $1 \leq SR$ and $RS \leq 1$. Then by modularity we have

$$1 = 1 \cap 1 \leq SR \cap 1 \leq (S \cap R^{\mathrm{op}})R \leq R^\circ R$$

and as $1 = 1^{op} \leq R^{\mathrm{op}}S^{\mathrm{op}}$ again by modularity we also have

$$1 \leq R^{\mathrm{op}}S^{\mathrm{op}} \cap 1 \leq (R^{\mathrm{op}} \cap S)S^{\mathrm{op}} \leq SS^{\mathrm{op}} \ .$$

Thus, as $RS \leq 1$ it follows that $R \leq RSS^{\mathrm{op}} \leq S^{\mathrm{op}}$ and $S \leq R^{\mathrm{op}}RS \leq R^{\mathrm{op}}$, i.e. $S^{\mathrm{op}} \leq R$, from which it follows that $S = R^{\mathrm{op}}$ and R is a map. $\square$

In [FrSc](2.1) it has been shown that an allegory $\mathcal{A}$ is equivalent to $\mathbf{R}el(\mathbf{C})$ for some regular category $\mathbf{C}$ iff $\mathcal{A}$ is unitary and tabular. [6] The central notion of *tabularity* is defined in [FrSc] as follows: for every $R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ there are maps $p : P \to A$ and $g : P \to B$ which *tabulate* R, i.e. $R = qp^{\mathrm{op}}$ and $q^{\mathrm{op}}q \cap p^{\mathrm{op}}p = 1$. In [FrSc] it has been shown that for maps p and q tabulating R it holds that for all maps $f : C \to A$ and $g : C \to B$ with $gf^{\mathrm{op}} \leq R$ there is a unique map $h : C \to P$ with $f = ph$ and $g = qh$. Thus, in a unitary tabular allegory $\mathcal{A}$ a pair of maps $p : P \to A$ and $q : P \to B$ tabulates

[6] Up to equivalence the $\mathbf{C}$ is given by $\mathrm{Map}(\mathcal{A})$, the category of maps in $\mathcal{A}$.

$R : A \longrightarrow\!\!\!\!\!\!+\;\; B$ iff $R = qp^{\mathrm{op}}$ and (p, q) is terminal in the category of spans (f, g) with $gf^{\mathrm{op}} \leq R$.

Thus, for categories of relations over a regular category the following holds.

Theorem 5.4 *For a regular category* $\mathbf{C}$ *for every relation* $R : A \longrightarrow\!\!\!\!\!\!+\;\; B$ *there exists maps* $p : P \to A$ *and* $q : P \to B$ *such that*

(1) $R = qp^{\mathrm{op}}$ *and*

(2) *whenever* $R = gf^{\mathrm{op}}$ *then there is a unique map* h *with* $f = ph$ *and* $g = qh$.

6 Distributors at Work

In this section we consider a couple of examples demonstrating how notions and facts of basic category theory can be reformulated and generalised using the conceptual framework of distributors. For ease of exposition in the sequel we will not distinguish notationally between $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\;\; \mathbf{B}$ and $\hat{\phi} : \mathbf{A} \to \widehat{\mathbf{B}}$.

6.1 Some simple examples

It is well known that for an adjunction $F \dashv U$ the functor F is full and faithful iff the unit η is a natural isomorphism. This generalises to arbitrary functors $F : \mathbf{A} \to \mathbf{B}$ between small categories without assuming that F has a right adjoint as from Lemma 5.1 we know that F is full and faithful iff the unit η of the adjunction $\phi_F \dashv \phi^F$ is an isomorphism.

For a functor $f : \mathbf{A} \to \mathbf{B}$ between small categories the change of base functor $f^* : \widehat{\mathbf{B}} \to \widehat{\mathbf{A}}$ has a left adjoint $f_!$ given by the left Kan extension of $\mathsf{Y}_{\mathbf{B}} \circ f : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\;\; \widehat{\mathbf{B}}$ along Yoneda as

$$f_! \circ \mathsf{Y}_{\mathbf{A}} = \mathsf{Y}_{\mathbf{B}} \circ f = \phi_f \dashv \phi^f = f^* \circ \mathsf{Y}_{\mathbf{B}}$$

Let I be a discrete category, i.e. a set, and $F : I \to \mathbf{A}$. The corresponding family $(A_i)_{i \in I}$ in $\mathbf{A}$ is *generating* iff the family of functors $\mathrm{Hom}(A_i, -) : \mathbf{A} \to \mathbf{Set}$ is collectively faithful, i.e. iff $\phi^F : \mathbf{A} \to \mathbf{Set}^{I^{op}}$ is faithful. Thus, we may define a functor $F : \mathbf{I} \to \mathbf{A}$ to be generating iff $\phi^F : \mathbf{A} \to \widehat{\mathbf{I}}$ is faithful.

Similarly, $F : I \to \mathbf{A}$ is a *strong family of generators* (*cf.* Definition 4.5.13 of vol.1 of [Bor]) iff $\phi^F : \mathbf{A} \to \mathbf{Set}^{I^{op}}$ reflects isomorphisms. Thus, we may define a functor $F : \mathbf{I} \to \mathbf{A}$ to be strongly generating iff $\phi^F : \mathbf{A} \to \widehat{\mathbf{I}}$ is faithful and reflects isomorphisms.

A full subcategory $I : \mathbf{D} \hookrightarrow \mathbf{A}$ is called a *dense family* iff every object in $\mathbf{A}$ arises as the (canonical) colimit of its approximating objects in $\mathbf{D}$. Proposition 4.5.13 of vol.1 of [Bor] says that $\mathbf{D}$ is a dense family iff for $\phi^I : \mathbf{A} \to \widehat{\mathbf{D}} : A \mapsto \mathbf{A}(I(-), A)$ is full and faithful. Accordingly, we may call a functor $F : \mathbf{B} \to \mathbf{A}$ dense iff $\phi^F : \mathbf{A} \to \widehat{\mathbf{B}}$ is full and faithful. Notice that F is stronly generating whenever F is dense as ϕ^F reflects isomorphisms whenever ϕ^F is full and faithful.

Moreover, from these considerations it follows easily that a category $\mathbf{A}$ is well–powered if it has a dense family $I : \mathbf{D} \hookrightarrow \mathbf{A}$ as in this case $\phi^I : \mathbf{A} \to \widehat{\mathbf{D}}$ is full and faithful and such functors reflect well–poweredness ($\widehat{\mathbf{D}}$ is well–powered as it is a presheaf topos).

6.2 Finality

A functor $F : \mathbf{A} \to \mathbf{B}$ is called final iff for every object B in $\mathbf{B}$ the comma category $B{\downarrow}\mathbf{F}$ is nonempty and connected. Obviously, a functor F is final iff $F_! : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ preserves terminal objects. Thus, final functors are closed under composition as if $F_!$ and $G_!$ preserve terminal objects then so does $(GF)_! \cong G_! F_!$. The identity functor $\mathrm{Id}_{\mathbf{A}}$ is always final but the Yoneda functor $\mathsf{Y}_{\mathbf{A}} : \mathbf{A} \to \widehat{\mathbf{A}}$ is not final if $\mathbf{A}$ is not connected (as then $1{\downarrow}\mathsf{Y}_{\mathbf{A}}$ is empty).

6.3 Flatness

On p.260 of [Bor] flatness is defined as follows.

(1) A functor $F : \mathbf{A} \to \mathbf{Set}$ is *flat* iff its category of elements $\mathbf{Elts}(F)$ is cofiltered.

(2) A functor $F : \mathbf{A} \to \mathbf{B}$ is *flat* iff for all $B \in \mathbf{B}$ the functor $\mathbf{B}(B, F(-)) : \mathbf{A} \to \mathbf{Set}$ is flat in the sense of (1).

First of all this definition is not well constructed as when instantiating $\mathbf{B}$ by $\mathbf{Set}$ in (2) one gets a condition which at first sight is stronger than condition (1), i.e. it needs some argument to see that they are equivalent. Thus, the right way of defining flatness of $F\mathbf{A} \to \mathbf{B}$ this way would be to require that $\mathbf{Elts}(\mathbf{B}(B, F(-))$ is cofiltered for all objects $B \in \mathbf{B}$. However, such a definition of flatness is inappropriate as it does not capture the idea of flatness. It rather should arise as an *a posteriori* characterisation of the appropriately defined notion of flatness. That is what we will do next.

Recall that for a functor $F : \mathbf{A} \to \mathbf{B}$ the left adjoint $F_! : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ to the "reindexing" functor $F^* : \widehat{\mathbf{B}} \to \widehat{\mathbf{A}}$ is given by $\mathrm{L}_{\mathbf{A}}(\mathsf{Y}_{\mathbf{B}} \circ F)$, and, therefore, isomorphic to $\phi_F(-) : \mathbf{Dist}(\mathbf{1}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{1}, \mathbf{B})$.

Usually, F is called flat iff $F_! : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ preserves finite limits, i.e. iff $\phi_F(-) : \mathbf{Dist}(\mathbf{1}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{1}, \mathbf{B})$ preserves finite limits. However, we can get rid of the somewhat unnatural restriction restriction to $\mathbf{1}$ by the following lemma.

Lemma 6.1 *For a distributor* $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ *the following properties are equivalent*

(1) $\phi(-) : \mathbf{Dist}(\mathbf{1}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{1}, \mathbf{B})$ *preserves finite limits*

(2) $\phi(-) : \mathbf{Dist}(\mathbf{C}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{C}, \mathbf{B})$ *preserves finite limits for all* $\mathbf{C}$.

Proof. Clearly, (2) implies (1). The reverse direction holds as (finite) limits in functor categories are pointwise. ☐

These considerations suggest the following definition of flatness of functors.

Definition 6.1 *A functor* $F : \mathbf{A} \to \mathbf{B}$ *is* flat *iff*

$$\phi_F(-) : \mathbf{Dist}(\mathbf{C}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{C}, \mathbf{B})$$

preserves finite limits for all $\mathbf{C}$.

One of the advantages of this reformulation of the notion of flatness is for example the simple proof of the following lemma.

Lemma 6.2 *Let* $F : \mathbf{A} \to \mathbf{B}$ *and* $G : \mathbf{B} \to \mathbf{C}$ *be functors where* G *is full and faithful. Then* F *is flat whenever* GF *is flat.*

Proof. As G is full and faithful we have $\mathrm{id}_{\mathbf{B}} \cong \phi^G \phi_G$ and accordingly $\phi_F \cong \phi^G \phi_G \phi_F$. As GF is flat by assumption $\phi_G \phi_F(-)$ preserves finite limits. As ϕ^G is right adjoint to ϕ_G we have that $\phi^G(-)$ preserves (finite) limits. Thus, $\phi^G \phi_G \phi_F(-)$ preserves finite limits, too. As $\phi_F \cong \phi^G \phi_G \phi_F$ it follows that $\phi_F(-)$ preserves finite limits, i.e. F is flat. □

Another advantage of our definition of flatness is that it easily generalises to distributors.

Definition 6.2 *A distributor $\phi : \mathbf{A} \to \mathbf{B}$ is flat iff $\phi(-) : \mathbf{Dist}(\mathbf{C}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{C}, \mathbf{B})$ preserves finite limits for all $\mathbf{C}$.*

The analogy with the notion of flatness for modules is apparent when thinking of composition of distributors as tensor product.

Notice that by Lemma 6.1 $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbf{B}$ is flat iff $\phi(-) : \mathbf{Dist}(\mathbf{1}, \mathbf{A}) \to \mathbf{Dist}(\mathbf{1}, \mathbf{B})$ preserves finite limits, i.e. iff $\mathrm{L}_{\mathbf{A}}(\phi) : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ preserves finite limits.

We now introduce the notions "filtered" and "cofiltered" which are most useful for characterising flatness in more elementary terms.

Definition 6.3 *A category $\mathbf{C}$ is* filtered *iff for every finite diagram in $\mathbf{C}$ there is a cocone over it and a category $\mathbf{C}$ is called* cofiltered *if its dual $\mathbf{C}^{op}$ is filtered, i.e. iff for every finite diagram in $\mathbf{C}$ there is a cone over it.*[7]

In [Bor] for example one can find the following characterisation of covariant presheaves whose Kan extension preserves finite limits.

Theorem 6.3 *For a functor $F : \mathbf{A} \to \mathbf{Set}$ its left Kan extension $\mathrm{L}_{\mathbf{A}}(F) : \widehat{\mathbf{A}} \to \mathbf{Set}$ preserves finite limits if and only if its category of elements $\mathbf{Elts}(F)$ is cofiltered.*

This gives rise to the following characterisation of flat functors.

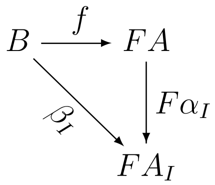
Theorem 6.4 *A functor $F : \mathbf{A} \to \mathbf{B}$ is flat iff $B{\downarrow}F$ is cofiltered for every object B in $\mathbf{B}$.*

---

[7]Note that some people use the word "cofiltered" instead of "filtered" e.g. in MacLane and Moerdijk's book *Sheaves in Geometry and Logic* whereas in MacLane's classical book *Categories for the Working Mathematician* "filtered" is defined as in our present Definition 6.3. Although this traditional terminology has the disadvantage that a poset is a filter iff its dual is filtered as a category we stick to it in our notes as it is the more common one in the literature.

Proof. As limits and colimits are pointwise and due to the construction of left Kan extensions we have that $\mathrm{L}_{\mathbf{A}}(\mathsf{Y}_{\mathbf{B}} \circ F) : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ preserves finite limits iff $\mathrm{L}_{\mathbf{A}}(\mathbf{B}(B, F(-)))$ preserves finite limits for all $B \in \mathbf{B}$. But the latter condition by Theorem 6.3 is equivalent to the requirement that $\mathbf{Elts}(\mathbf{B}(B, F(-)))$ is cofiltered for all $B \in \mathbf{B}$. As $\mathbf{Elts}(\mathbf{B}(B, F(-))) \cong B{\downarrow}F$ we get that $F : \mathbf{A} \to \mathbf{B}$ is flat iff $B{\downarrow}F$ is cofiltered. $\square$

One easily checks that $B{\downarrow}F$ is cofiltered iff for every finite diagram $D : \mathbb{D} \to \mathbf{A}$ and every cone $\beta : \Delta(B) \Rightarrow FD$ there is a cone $\alpha : \Delta(A) \Rightarrow D$ such that β factors through $F\alpha$, i.e. there is a map $f : F(A) \to B$ with



for all $I \in \mathbb{D}$.

The characterisation of flatness given in Theorem 6.4 extends to distributors in the following way.

Theorem 6.5 *A distributor* $\phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ *is flat iff* $B{\downarrow}\phi$ *is cofiltered for all* $B \in \mathbf{B}$ *where* $B{\downarrow}\phi$ *stands for* $\mathbf{Elts}(\phi(B, -))$.

Proof. First recall that ϕ is flat iff $\mathrm{L}_{\mathbf{A}}(\phi) : \widehat{\mathbf{A}} \longrightarrow\!\!\!\!\!\!\!+\;\; \widehat{\mathbf{B}}$ preserves finite limits. As limits and colimits are pointwise in presheaf categories and due to the construction of left Kan extensions we have that $\mathrm{L}_{\mathbf{A}}(\phi)$ preserves finite limits iff all $\mathrm{L}_{\mathbf{A}}(\phi(B, -))$ preserve finite limits. By Theorem 6.3 this is equivalent to $B{\downarrow}\phi = \mathbf{Elts}(\phi(B, -))$ being cofiltered for all $B \in \mathbf{B}$. $\square$

Notice that Theorem 6.4 follows from Theorem 6.5 since $B{\downarrow}F \cong B{\downarrow}\phi_F$.

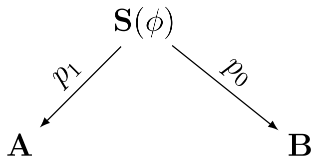
As we shall see in the next subsection flatness of a distributor will turn out as equivalent to flatness of certain functors associated with a distribution in a canonical way.

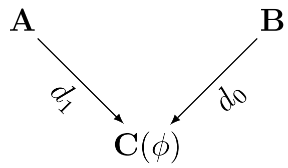
We suggest to try the following

Exercise. Show that $F : \mathbf{A} \to \mathbf{B}$ is dense iff $\mathrm{Hom}^{\mathbf{A}}(\phi^F, \phi^F) \cong \mathrm{id}_{\mathbf{B}}$ (Hint: Use that F is dense iff ϕ^F is full and faithful!).

6.4 Distributors, Comma and Cocomma Categories

In this section we will show how a distributor $\phi : \mathbf{A} \looparrowright \mathbf{B}$ can be factorised as $\phi \cong \phi_{p_0}\phi^{p_1}$ and $\phi \cong \phi^{d_0}\phi_{d_1}$ where





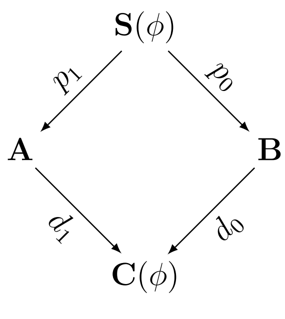
are canonically chosen spans and cospans whose construction we describe next.

The category $\mathbf{C}(\phi)$ is constructed as follows. Its objects are given by the disjoint union of the objects of $\mathbf{A}$ and $\mathbf{B}$, i.e. $\mathrm{Ob}(\mathbf{C}(\phi)) = \mathrm{Ob}(\mathbf{A}) \amalg \mathrm{Ob}(\mathbf{B})$. The morphism of $\mathbf{C}(\phi)$ are the morphisms of $\mathbf{A}$ and $\mathbf{B}$ together with "formal arrows" $x : B \to A$ for every $x \in \phi(B, A)$. Composition is inherited from $\mathbf{A}$ and $\mathbf{B}$, respectively, for pairs of arrows living in the same component. For formal arrows $x : B \to A$ and $a : A \to A'$, $b : B' \to B$ in $\mathbf{A}$ and $\mathbf{B}$, respectively, we put

$$ax = \phi(B, a)(x) \qquad \text{and} \qquad xb = \phi(A, b)(x)$$

in accordance with notation introduced earlier. We write $d_0 : \mathbf{B} \hookrightarrow \mathbf{C}(\phi)$ and $d_1 : \mathbf{A} \hookrightarrow \mathbf{C}(\phi)$ for the respective full embeddings. Then one easily verifies that $\phi \cong \phi^{d_0}\phi_{d_1}$ which factorisation is called the *cospan representation of* ϕ. This assignment of cospans to distributors extends to a functor $\mathbf{C} : \mathbf{Dist}(\mathbf{A}, \mathbf{B}) \to \mathbf{Cospan}(\mathbf{A}, \mathbf{B})$ by sending a natural transformation $\varphi : \phi \Rightarrow \psi$ to the functor $\mathbf{C}(\varphi) : \mathbf{C}(\phi) \to \mathbf{C}(\psi)$ which maps a formal arrow $x \in \phi(B, A)$ to the formal arrow $\varphi_{B,A}(x) \in \psi(B, A)$ and behaves like identity on objects and all other arrows.

The corresponding *span representation of* ϕ is obtained by taking the (non-commuting) comma square



where $\mathbf{S}(\phi) = d_0{\downarrow}d_1$ and p_0 and p_1 are the source and target functor, respectively, for which one easily shows that $\phi \cong \phi_{p_0}\phi^{p_1}$. The category $\mathbf{S}(\phi)$ can be described more elementarily as follows. Its objects are given by the disjoint union of the $\phi(B, A)$, i.e. $\mathrm{Ob}(\mathbf{S}(\phi)) = \coprod_{B,A} \phi(B, A)$. A morphism from $x \in \phi(B, A)$ to $x' \in \phi(B', A')$ is a pair (b, a) where $a : A \to A'$ and $b : B \to B'$ such that $x'b = ax$. Composition of morphisms is componentwise and inherited from $\mathbf{A}$ and $\mathbf{B}$. Again this assignment of spans to distributors extends to a functor $\mathbf{S} : \mathbf{Dist}(\mathbf{A}, \mathbf{B}) \to \mathbf{Span}(\mathbf{A}, \mathbf{B})$ by sending a natural transformation $\varphi : \phi \Rightarrow \psi$ to the functor $\mathbf{S}(\varphi) : \mathbf{S}(\phi) \to \mathbf{S}(\psi)$ which maps a formal arrow $x \in \phi(B, A)$ to the formal arrow $\varphi_{B,A}(x) \in \psi(B, A)$ and behaves like identity on objects and all other arrows.

These constructions satisfy the following important

Theorem 6.6 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be categories. Then the functors*

$$\mathbf{C} : \mathbf{Dist}(\mathbf{A}, \mathbf{B}) \to \mathbf{Cospan}(\mathbf{A}, \mathbf{B}) \quad \textit{and} \quad \mathbf{S} : \mathbf{Dist}(\mathbf{A}, \mathbf{B}) \to \mathbf{Span}(\mathbf{A}, \mathbf{B})$$

are both full and faithful. Moreover, the functor $\mathbf{C}$ *has a right adjoint sending cospan* (i, j) *to* $\phi^j\phi_i$ *and the functor* $\mathbf{S}$ *has a left adjoint sending span* (p, q) *to* $\phi_q\phi^p$.

Proof. Left as an exercise. □

As a consequence we get that $\mathbf{Dist}(\mathbf{A}, \mathbf{B})$ is

(1) equivalent to the full reflective subcategory of $\mathbf{Span}(\mathbf{A}, \mathbf{B})$ of those spans $\mathbf{A} \xleftarrow{p} \mathbf{C} \xrightarrow{q} \mathbf{B}$ where p is a cofibration, q is a fibration and $\langle p, q\rangle : \mathbf{C} \to \mathbf{A}{\times}\mathbf{B}$ has discrete fibres and

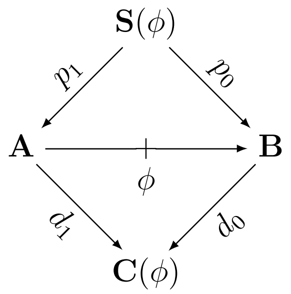
(2) equivalent to the full reflective subcategory of $\mathbf{Cospan}(\mathbf{A}, \mathbf{B})$ of those cospans $\mathbf{A} \xrightarrow{i} \mathbf{C} \xleftarrow{j} \mathbf{B}$ where i is a cosieve inclusion, j is a sieve inclusion and each object of $\mathbf{C}$ is either of the form $i(A)$ or of the form $j(B)$.

Notice that for a cospan $\mathbf{A} \xrightarrow{f} \mathbf{C} \xleftarrow{g} \mathbf{B}$ its comma span is given by $\mathbf{S}(\phi^g\phi_f)$ and, therefore, cospans give rise to the same comma span if and only if they induce the same distributor.

Accordingly, for a span $\mathbf{A} \xleftarrow{f} \mathbf{C} \xrightarrow{g} \mathbf{B}$ its cocomma cospan is given by $\mathbf{C}(\phi_g \phi^f)$ and, therefore, spans give rise to the same cocomma cospan if and only if they induce the same distributor.

We spend the rest of this subsection to the discussion of flatness criteria for distributors arising from their canonical span and cospan representations.

Lemma 6.7 *For a distributor $\phi : \mathbf{A} \nrightarrow \mathbf{B}$ with canonical span and cospan representations*



ϕ is flat iff d_1 is flat iff p_0 is flat.

Proof. Notice that $B{\downarrow}d_1 \cong B{\downarrow}\phi$ for all $B \in \mathbf{B}$ and $A{\downarrow}d_1 \cong A{\downarrow}\mathbf{A}$ is cofiltered for all $A \in \mathbf{A}$. Thus, $B{\downarrow}\phi$ is cofiltered for all $B \in \mathbf{B}$ iff $C{\downarrow}d_1$ is cofiltered for all $C \in \mathbf{C}$. Thus, by Theorems 6.4 and 6.5 it follows that ϕ is flat iff d_1 is flat.

The functor p_0 is a fibration whose fibre over B is $B{\downarrow}d_1$. But one knows that a fibration is flat iff all its fibres are cofiltered. Thus, the functor p_0 is flat iff all its fibres $B{\downarrow}d_1$ are cofiltered. Thus, by Theorem 6.4 it follows that p_0 is flat iff d_1 is flat. $\square$

Though the previous lemma does not extend to arbitrary non-canonical span and cospan representations one nevertheless has the following sufficient criteria: a distributor $\phi_q \phi^p$ is flat if q is flat (as the right adjoint ϕ^p is flat anyway) and a distributor $\phi = \phi^j \phi_i$ is flat if the functor i is flat (as the right adjoint ϕ^j is flat anyway).

6.5 Abstract Kan Extensions

Next we investigate Kan extensions within bicategories.

Definition 6.4 *Let $\mathfrak{B}$ be a bicategory. A* 1*-cell $f : A \to B$ in $\mathfrak{B}$ is called a* left/right Kan map *iff*

$$\mathfrak{B}(f, C) : \mathfrak{B}(B, C) \to \mathfrak{B}(A, C)$$

has a left/right adjoint L_f for all objects C.

If $\mathfrak{B}$ is the category **Bim** of bimodules or the category **Dist** of distributors then $\mathfrak{B}(f, C)$ always has a right adjoint as it is cocontinuous.[8] However, $\mathfrak{B}(f, C)$ need not always have a left adjoint as composition does not preserve limits in general as there are modules and distributors which are not flat!

Generally, in bicategories $\mathfrak{B}$ a 1–cell $f : A \to B$ is a left Kan map if f has a right adjoint in $\mathfrak{B}$. In the special case of **Dist**, however, it turns out that ϕ is a left Kan map iff ϕ has a right adjoint distributor, i.e. iff ϕ is essentially a functor (*cf.* Theorem 5.2).

By the adjoint functor theorem a distributor ϕ is a left Kan map in **Dist** iff $(-)\phi$ preserves all limits, i.e. iff $\phi^\sigma(-)$ preserves all limits. A distributor ϕ is usually called *absolutely flat* iff $\phi(-)$ preserves *all* limits. Thus, ϕ is a left Kan map in **Dist** iff ϕ^σ is absolutely flat.

7 Distributors and Generalised Fibrations

In this final section we describe how via distributors the notion of a fibred category can be generalised considerably. This is only the beginning of a long story which may be told at another place in more detail.

It was noticed by the author back in the early 70ies that with an arbitrary functor $P : \mathbf{X} \to \mathbf{I}$ one may associate a normalised[9] lax functor

$$dP : \mathbf{I}^{op} \to \mathbf{Dist}$$

where $dP(I)$ is the fibre $P(I)$ of P over I and for $\alpha : J \to I$ in $\mathbf{I}$ the distributor $dP(\alpha) : P(I) \longrightarrow\!\!\!\!\!\!\!+\;\; P(J)$ is given by

$$dP(\alpha)(Y, X) = \{f : Y \to X \mid P(f) = \alpha\}$$

$$dP(\alpha)(b, a)(f) = afb \quad .$$

[8]Notice that in these bicategories $\mathfrak{B}(C, f) : \mathfrak{B}(C, A) \to \mathfrak{B}(C, B)$ is always cocontinuous and, therefore, has a right adjoint called *right coextension along f*.

[9]meaning here that identities are preserved by the lax functor

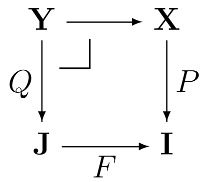
Moreover, every functor F from P to $Q : \mathbf{Y} \to \mathbf{I}$ over $\mathbf{I}$ (i.e. $QF = P$) gives rise to a lax natural transformation $dF : dP \Rightarrow dQ$ whose components are the fibres F_I of F.

On the other hand with every normalised lax functor $D : \mathbf{I}^{op} \to \mathbf{Dist}$ one may associate via (a slightly adapted) Grothendieck construction a functor $P : \int D \to \mathbf{I}$ and these processes of "differentiation" and "integration" are mutually inverse to each other.

There arises the question to which extent properties of a functor $P : \mathbf{X} \to \mathbf{I}$ can be expressed equivalently in terms of the normalised lax functor $dP : \mathbf{I}^{op} \to \mathbf{Dist}$. Actually this is possible for quite a few examples some of which will be discussed in this section.

For example a functor P is a prefibration iff dP factors through $\mathbf{Cat}$ and it is a fibration iff moreover dP is a pseudofunctor.

This suggest that *for any subbicategory* $\mathfrak{B}$ *of* $\mathbf{Dist}$ *we may consider those functors* P *whose derivative* dP *factors through* $\mathfrak{B}$. Obviously, such a class of functors is stable under pullbacks along arbitrary functors as for any pullback



in $\mathbf{Cat}$ it holds that

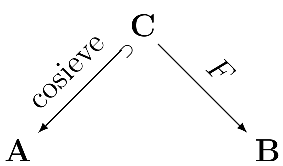
$$dQ \simeq dP \circ F$$

and, therefore, dQ factors through $\mathfrak{B}$ whenever P factors through $\mathfrak{B}$.

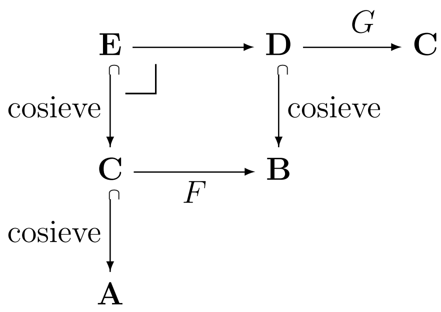
Just to mention a further interesting example we may take for $\mathfrak{B}$ the subbicategory of flat distributors.

Another example is the subbicategory $\mathfrak{B}$ of $\mathbf{Dist}$ containing all categories as objects and whose 1–cells are the *partial functors* which will be defined in a moment and which contains all 2–cells between partial functors.

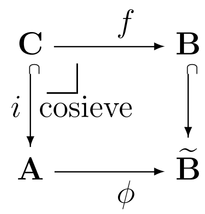
A partial functor from $\mathbf{A}$ to $\mathbf{B}$ is given by a span



whose left leg is the inclusion of a cosieve. As cosieve inclusions are closed under composition and stable under pullbacks along arbitrary functors partial functors can be composed as follows



in accordance with the usual definition of composition for partial maps. Notice that partial functors are up to isomorphism in 1–1–correspondence with those distributors $\phi : \mathbf{A} \to \widehat{\mathbf{B}}$ which factor through $\widetilde{\mathbf{B}}$, the full subcategory of $\widehat{\mathbf{B}}$ of representable presheaves and 0, the empty presheaf. More explicitely, this 1–1–correspondence sends a partial functor $(i : \mathbf{C} \hookrightarrow \mathbf{A}\,, f : \mathbf{A} \to \mathbf{B})$ to the distributor $\phi_f \phi^i$ and a $\phi : \mathbf{A} \to \widetilde{\mathbf{B}}$ to the span (i, f) as given by the pullback



where $\mathbf{B} \hookrightarrow \widetilde{\mathbf{B}}$ is the inclusion of representable presheaves.[10] One readily checks that this 1–1–correspondence respects composition of these (particular) spans and these (particular) distributors. Notice, however, that the partial functor corresponding to a distributor $\phi : \mathbf{A} \to \widetilde{\mathbf{B}} \hookrightarrow \widehat{\mathbf{B}}$ as described by the pullback above is different already in trivial cases from the terminal span representing the distributor ϕ . Consider e.g. the identity on $\mathbf{A}$ as given by $\mathsf{Y}_{\mathbf{A}} : \mathbf{A} \to \widetilde{\mathbf{A}} \hookrightarrow \widehat{\mathbf{A}}$ then the corresponding partial functor is given by the

[10]Notice that the inclusion $\mathbf{1} \hookrightarrow \widetilde{\mathbf{1}}$ classifies cosieve inclusions in **Cat**. In this sense $\mathbf{1} \hookrightarrow \widetilde{\mathbf{1}}$ resembles the subobject classifiers of toposes. On the other hand the inclusions $\mathbf{B} \hookrightarrow \widetilde{\mathbf{B}}$ resemble the partial map classifiers of toposes. This analogy explains why we have chosen the notation $\widetilde{\mathbf{B}}$.

span $(\mathrm{id}_{\mathbf{A}}, \mathrm{id}_{\mathbf{A}})$ whereas the terminal span representing $\mathsf{Y}_{\mathbf{A}}$ is given by

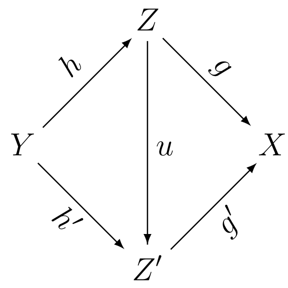
$$\begin{array}{ccc} & \mathbf{A}^{\mathbf{2}} & \\ {}^{\partial_0}\swarrow & & \searrow^{\partial_1} \\ \mathbf{A} & & \mathbf{A} \end{array}$$

which coincide if and only if $\mathbf{A}$ is discrete.

The functors whose derivative factors through the subbicategory of partial functors can be characterised as those functors where cartesian arrows are closed under composition but cartesian liftings of X along some α exist if and only if there exists some lifting of X along α. Alternatively, one may characterise them as those functors $P : \mathbf{X} \to \mathbf{I}$ whose class $\mathbf{V}$ of vertical arrows forms part of a *foliation* on $\mathbf{X}$.

Another important class of *generalised* or *weak* fibrations are the so–called *homotopy fibrations* $P : \mathbf{X} \to \mathbf{I}$ where the category of liftings of X (in $P(I)$) along $\alpha : J \to I$ in $\mathbf{I}$ is always nonempty and connected. They can be characterised as those functors whose derivative factors through the subbicategory of $\mathbf{Dist}$ consisting of those distributors $\phi : \mathbf{A} \to \widehat{\mathbf{B}}$ where ϕ factors through the full subcategory of $\widehat{\mathbf{B}}$ on those presheaves P whose category of elements $\mathbf{Elts}(P)$ is nonempty and connected. Dropping the requirement of connectedness but keeping the requirement of nonemptyness gives rise to the important class of *Serre fibrations* consisting of those functors $P : \mathbf{X} \to \mathbf{I}$ where for every $\alpha : J \to I$ and X over I there is a morphism $f : Y \to X$ over α.

The functors $P : \mathbf{X} \to \mathbf{I}$ whose derivative $dP : \mathbf{I}^{op} \to \mathbf{Dist}$ is a *pseudo–* functor and not just a lax normalized functor are called *Conduché fibrations*. Thus P is a fibration if and only if P is a Conduché fibration and a prefibration. Conduché fibrations have been characterised (independently by J. Giraud and F. Conduché) as those functors $F : \mathbf{A} \to \mathbf{B}$ for which the change of base functor $F^* : \mathbf{Cat}/\mathbf{B} \to \mathbf{Cat}/\mathbf{A}$ has a right adjoint $\prod_F$. More elementarily $P : \mathbf{X} \to \mathbf{I}$ is a Conduché fibration iff for every $f : Y \to X$ in $\mathbf{X}$ and $\beta : J \to K$ and $\alpha : K \to I$ with $\alpha \circ \beta = P(f)$ the category $\mathsf{Split}_{\alpha,\beta}(f)$ is connected and nonempty where the morphisms of $\mathsf{Split}_{\alpha,\beta}(f)$ are commuting diagrams



with $g \circ h = f = g' \circ h'$, $P(g) = \alpha = P(g')$, $P(h) = \beta = P(h')$ and u vertical.

One easily may imagine that there is a lot of other interesting examples in this vein which are worthwhile to be investigated.

References

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