Forcing Axioms: Tears and Smiles

Rahman Mohammadpour

Some people are always critical of vague statements. I tend rather to be critical of precise statements; they are the only ones which can correctly be labelled wrong.
(Raymond Smullyan)

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- A sentence s is said to be **independent from a theory** T if both $T \cup \{s\}$ and $T \cup \{\neg s\}$ are consistent theories.

A couple of basic questions:

1. When a theory T is incomplete (i.e., there are independent statements)?
2. Whether we can extend a consistent theory T to a complete consistent theory T^* (i.e., T^* is consistent and for every sentence s, either $T^* \vdash s$ or $T^* \vdash \neg s$)?

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Theorem (Lindenbaum) Every first-order consistent theory in a countable language has a complete extension.

$$
\begin{tikzcd}[column sep=small]
\cdots & \cdots & & & \cdots & \cdots \\
T \cup \{\neg s_1\} \cup \{\neg s_2\} & T \cup \{\neg s_1\} \cup \{s_2\} & & T \cup \{s_1\} \cup \{\neg s_2\} & T \cup \{s_1\} \cup \{s_2\} & \\
& T \cup \{\neg s_1\} \arrow[ul] \arrow[u] & & T \cup \{s_1\} \arrow[u] \arrow[ur] & & \\
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\end{tikzcd}
$$

If T^* is obtained by a cofinal branch through the above tree, then T^* is consistent and complete and has the following trivial property:

$$T^* \vdash s \Rightarrow T \nvdash \neg s.$$

Mathematics

- Language: $\in$
- Logic: the first order logic.
- Theory: ZFC.
- By Gödel's Incompleteness Theorem, ZFC is incomplete.

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or at least,

$$T){-}_{\mathfrak{m}}\phi \Rightarrow T(\mathfrak{m}) \vdash \phi$$

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Unfortunately, this is not always possible without limitations. However, there are several pseudo-completions in set theory:

- generic absoluteness,
- modal maximality principles introduced by Stavi and Väänänen and by Hamkins,
- inner models,
- **forcing axioms**, etc.

We can now forget about the beginning quotation!

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The method of forcing was invented by Paul Cohen in 1963 to show that the Continuum Hypothesis is not provable in mathematics. Combined with an earlier result due to Gödel, it was shown that the Continuum Hypothesis is independent from ZFC.

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The Continuum Hypothesis (CH) states that the size of the real line $\mathbb{R}$ is the first uncountable cardinal ω_1.

Forcing, as a method, is a process as follows:

- Start with a finite fragment of ZFC, say T.
- Pick a model M of a finite fragment S of ZFC with $S \supseteq T$.
- Adjoin a generic object G to M, by an internal device, say $M[G]$.
- so that $M[G]$ is a model of T.

How does this process help us in independence?

- We assume ZFC is consistent, and we want to show that s is consistent with ZFC.
- If this not the case, then there is a finite proof for $\neg s$ from ZFC, which uses a finite fragment of ZFC, say T,
- we may enlarge T to S so that we can perform the method of forcing internally,
- we can find a (countable and transitive) model M of S,
- the forcing is designated so that $\neg s$ holds in $M[G]$

Existence, or non-existence, that is the question

1. Why such an M exists?
2. What is G?
3. When G exists?
4. How to construct $M[G]$?
5. Why everything works at all?

A simulation

We want to know that whether the theory T_{fields} of fields implies that the equation $x^2 - 2 = 0$ has a solution, or logically whether $T_{\text{fields}} \vdash \exists x(x^2 - 2 = 0)$.

- We know that the statement $s \equiv$ "$\exists x\ x^2 - 2 = 0$" is independent from T_{fields} as $\mathbb{Q} \models \neg s$ and $\mathbb{R} \models s$, but
- we also know that we can make the field $\mathbb{Q}(\sqrt{2})$ which is the minimal extension of $\mathbb{Q}$, in which s is true.

Topology as a means to approximate points

Recall that a topological space $\mathbb{X}$ is a pair (X, τ) such that:

1. $\emptyset, X \in \tau \subseteq \mathcal{P}(X)$
2. τ is closed under arbitrary unions.
3. τ is closed under finite intersections.

The elements of τ are called **open** sets, which gauge the proximity between points in $\mathbb{X}$.

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- A set $D \subseteq X$ is **dense** if for every $U \in \tau$, $D \cap U \neq \varnothing$, i.e., D contains arbitrary approximations of any point.

BCT

- There are several versions of the Baire Category Theorem in mathematics.
- They are used to prove fundamental theorems in mathematical analysis.

Baire Category Theorem

Suppose $\mathbb{X}$ is a locally compact Hausdorff space. Then every countable collection $\{U_n : n \in \mathbb{N}\}$ of open dense subsets of $\mathbb{X}$ has non-empty intersection.

The BCT as a compactness phenomenon

The BCT can be used to show:

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2. Every two dense linearly ordered countable sets without end points are isomorphic.

• (compactness) Given a property P, if sufficiently many sub-objects of an object has the property P, then that object also has the property P.

BCT

Let $\mathbb{X}$ be a topological space.

Definition ($\mathrm{BCT}(\mathbb{X})$)

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Definition (Higher $\mathrm{BCT}(\mathbb{X})$)

The $\mathrm{BCT}(\mathbb{X}, \kappa)$ holds if every κ-sized collection $\mathcal{D}$ of open dense subsets of $\mathbb{X}$ has non-empty intersection.

A better way to approximate objects

A partially ordered set (henceforth poset) $\mathbb{P}$ is a pair $(\mathbb{P}, \leq)$, where $\leq$ is a binary relation such that

- for every $x \in \mathbb{P}$, $x \leq x$,
- for every $x, y, z \in \mathbb{P}$, if $x \leq y$ and $y \leq x$, then $x = y$,
- for every $x, y, z \in \mathbb{P}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

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• When $x \leq y$, we say that x is stronger than y. Intuitively, x is more informative than y.

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- A nonempty set $F \subseteq \mathbb{P}$ is called a **filter**, if F has the following properties:
  1. $\forall p \in F$ and $\forall q \in \mathbb{P}$, if $p \leq q$, then $q \in F$.
  2. $\forall p, q \in F$, there is $r \in F$ such that $r \leq p, q$.

When a filter is large?

A filter may be small, but...

Genericity

Suppose $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, a filter G is $\mathcal{D}$-generic if for every $D \in \mathcal{D}$, we have $G \cap D \neq \varnothing$.

Theorem (Rasiowa–Sikorski)

For every poset $\mathbb{P}$ and every countable collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter.

Definition (Higher RST)

For every poset $\mathbb{P}$ and a cardinal κ, $\mathrm{RST}(\mathbb{P}, \kappa)$ holds if for every κ-sized family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter.

Not a surprise!

- For every compact Hausdorff space $\mathbb{X}$, there is a poset $\mathbb{P}_{\mathbb{X}}$ such that

$$\mathrm{RST}(\mathbb{P}_{\mathbb{X}}, \kappa) \Rightarrow \mathrm{BCT}(\mathbb{X}, \kappa).$$

- For every poset $\mathbb{P}$, there is a compact Hausdorff space $\mathbb{X}_{\mathbb{P}}$ such that

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What is a forcing?

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We have the following simplified picture:

- We start with a (transitive) universe of sets V,
- we have a forcing $\mathbb{P} \in V$,
- we can find a filter $G \subseteq \mathbb{P}$ which meets all dense subsets of $\mathbb{P}$ which belong to V,
- we construct $V[G]$, which is a (transitive) model of ZFC.

An example

- $\mathbb{C} = \{p : A \to \{0,1\} : A \subseteq \mathbb{N} \text{ is finite}\}$ with $p \leq q$ if and only if p extends q as a function.

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- If G is V-generic, then g is a total function different from all functions from $\mathbb{N}$ into $\{0,1\}$ (in V.)
- Thus ZFC does NOT prove $\mathrm{RST}(\mathbb{C}, \omega_1)$! What about the negation?
- On the other hand, there are posets $\mathbb{P}$ such that ZFC disproves $\mathrm{RST}(\mathbb{P}, \omega_1)$!

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‡ It is easily seen that if $\mathbb{P}$ is a σ-closed forcing then $\mathrm{RST}(\mathbb{P}, \omega_1)$ holds true, and one can use this to show that $2^{\aleph_1} > \aleph_1$.

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- Some forms of the Baire Category Theorem are also relevant to general questions and follow from ZFC.
- The Rasiowa-Sikorski property of posets is equivalent to the Baire Category property of some spaces.
- They potentially imply compactness results.
- Some generalisations of the usual Rasiowa-Sikorski Theorem for various forcings and cardinals may be admissible candidates to pseudo-completions of ZFC

Forcing Axioms

Forcing Axiom

The forcing axiom $\mathrm{FA}(\mathfrak{K}, \kappa)$, for a class $\mathfrak{K}$ of posets and a cardinal κ, states that for every $\mathbb{P}$, and every sequence $\mathcal{D} = \langle D_\alpha : \alpha < \kappa \rangle$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter.

$\mathrm{FA}(\mathfrak{K}, \kappa)$ says, roughly speaking, if we can construct a set by meeting at most κ many dense sets, then such a set already exists!

A large amount of contradictory data may lead to destruction!

A set $A \subseteq \mathbb{P}$ is called an **antichain**, if for every $p \neq q$ in A, p and q are incompatible, i.e., there is no $r \in \mathbb{P}$ with $r \leq p, q$.

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A forcing $\mathbb{P}$ has the **countable chain condition (c.c.c.)** if and only if every antichain in $\mathbb{P}$ is at most countable.

Martin Axiom

MA_κ

Let MA_κ denote $\mathrm{FA}(CCC, \kappa)$.

Basic properties

1. MA_ω follows from the Rasiowa-Sikorski Theorem.
2. $\mathrm{MA}_{|\mathbb{R}|}$ is false.
3. MA_κ implies $|\mathbb{R}| > \kappa$.

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- MA states that for every $\kappa < |\mathbb{R}|$, MA_κ holds.

Theorem (Martin–Solovay, 1970)

MA is consistent with all possible values of $|\mathbb{R}|$.

How to obtain forcing axioms?

- A single forcing adds a generic object over the universe,
- we can iterate the process as long as we want,
- the iteration process can be carried out using a single poset which is called iterated forcing,
- the question is that whether we can add all possible generic objects
- specifically, whether we can guarantee that the question under consideration is the same as before,
- more specifically, whether we can guarantee the relevant cardinals remain cardinals.

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proceed as follows:

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• Solovay and Tennenbaum showed that the finite support iteration of c.c.c forcings has the c.c.c.

Generalisations: Teeter-totter

There are two other ways to generalise MA_{ω_1}:

1. (teeter!) increasing the number of dense sets to, say, ω_2 and changing the class of posets.
2. (totter!) keeping ω_1, while enlarging the class of c.c.c forcings.

- Replacing c.c.c forcings with ω_2-c.c. forcings fail. Let $\mathbb{P}$ be the poset of finite injections from ω_1 into $\mathbb{N}$.
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- But even then, Shelah has shown that the continuum hypothesis and $2^{\aleph_1} > \aleph_2$ implies the failure of $\mathrm{RST}(\mathbb{P}, \omega_2)$, for some σ-closed $\aleph_2$-c.c forcing $\mathbb{P}$.

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- There are some weak forcing axioms for subclasses of σ-closed $\aleph_2$-c.c. forcings! Why weak?

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PFA

The Proper Forcing Axiom (PFA) is the forcing axiom $\mathrm{FA}(\mathrm{Proper}, \omega_1)$.

- PFA is consistent relative to the existence of a supercompact cardinals. (Baumgartner)
- The first consistency result of PFA used a countable support iteration of proper forcings (due to Shelah).
- PFA implies $|\mathbb{R}| = \omega_2$. (Veličković, 1992)

MM

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Martin's Maximum (MM)

MM is the forcing axiom $\mathrm{FA}(\mathrm{sta.pres.}, \omega_1)$.

- MM is consistent relative to the consistency of a supercompact cardinal. (Foreman–Magidor–Shelah, 1988)

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Mathematics develops at limits

In 2014, Itay Neeman introduced a new method to iterate proper forcings using finite conditions. He reproved the consistency of PFA with his method. This was totally unexpected as the usual iteration with finite supports does not work.

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Let us leave the following question to the future.

Is Neeman's method a paradigm shift?

Moral conclusion

Our knowledge about ω_2 (or ω_3) is too narrow!

Truth is vulnerable!

... Applicable to thy case is the story of that fox which people saw running away in violent trepidation. Some one said to him "What calamity has happened to cause thee so much alarm?" He replied, "I have heard they are going to impress [to make fun of] the camel." They rejoined, "Oh Shatter-brain! What connection has a camel with thee, and what resemblance hast thou to it?" He answered, " Peace! for if the envious should, to serve their own ends, say, 'this is a camel' and I should be taken, who would take care about my release so as to inquire into my condition? and before the antidote brought from Irak the person who is bitten by the snake may be dead."

- Saadi Shirazi, Golestān, translated by Edward B. Eastwick

Thank you!